

# Saturation of Two Level Systems and Charge Noise in Josephson Junction Qubits

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We study the charge noise  $S_Q$  in Josephson qubits produced by fluctuating two level systems (TLS) with electric dipole moments in the substrate. The TLS are driven by an alternating electric field of angular frequency  $\Omega$  and electromagnetic energy flux  $J$ . It is not widely appreciated that TLS in small qubits can easily be strongly saturated if  $J \gg J_c$ , where  $J_c$  is the critical electromagnetic flux. To investigate the effect of saturation on the charge noise, we express the noise spectral density in terms of density matrix elements. To determine the dependence of the density matrix elements on the ratio  $J/J_c$ , we derive the steady state solution for the density matrix using the Bloch-Redfield differential equations. We then obtain a general expression for the spectral density of charge fluctuations as a function of frequency  $f$  and the ratio  $J/J_c$ . We find  $1/f$  charge noise at low frequencies, and that the charge noise is white (constant) at high frequencies. Using a flat density of states, we find that TLS saturation has no effect on the charge noise at either high or low frequencies.

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## I. INTRODUCTION

Noise and decoherence are a major obstacle to using superconducting Josephson junction qubits to construct quantum computers. Recent experiments<sup>1,2</sup> indicate that a dominant source of decoherence is two level systems (TLS) in the insulating barrier of the tunnel junction as well as in the dielectric material used to fabricate the circuit. It is believed that these TLS fluctuators produce charge noise  $S_Q(\omega)$ .<sup>3-10</sup>

However, previous theories of charge noise<sup>8-10</sup> have neglected the important issue of the saturation of the two level systems by electromagnetic radiation used to manipulate the qubits. Dielectric (ultrasonic) experiments on insulating glasses at low temperatures have found that when the electromagnetic (acoustic) energy flux  $J$  used to make the measurements exceeds the critical flux  $J_c$ , the dielectric (ultrasonic) power absorption by the TLS is saturated, and the attenuation decreases.<sup>2,11-14</sup> Previous theories of charge noise in Josephson junctions assumed that the TLS were not saturated, i.e., that  $J \ll J_c$ . This seems sensible since charge noise experiments<sup>15</sup> have been done in the limit where the qubit absorbed only one photon. However, stray electric fields could saturate TLS in the dielectric substrate as the following simple estimate shows. We can estimate the voltage  $V$  across the capacitor associated with the substrate and ground plane beneath a Cooper pair box (see Fig. 1) by setting  $CV^2/2 = \hbar\omega$  where  $\hbar\omega$  is the energy of the microwave photon. We estimate the capacitance  $C = \epsilon_o\epsilon_r A/L \sim 7$  aF using the area  $A = 40 \times 800$  nm<sup>2</sup> of the Cooper pair box, the thickness  $L = 400$  nm of the substrate,<sup>15</sup> and the relative permittivity  $\epsilon_r = 10$ . Using  $\omega/2\pi = f = 10$  GHz, we obtain a voltage of  $V \sim 1.4$  mV. A substrate thickness  $L$  of 400 nm yields an electric field of  $\mathcal{E} \sim 3.4 \times 10^3$  V/m. For amorphous SiO<sub>2</sub> at  $f = 7.2$  GHz and SiN<sub>x</sub> at  $f = 4.7$  GHz, the critical rms voltage  $V_c \sim 0.2$   $\mu$ V,<sup>2</sup> and

with a capacitor thickness of 300 nm, the critical field is  $\mathcal{E}_c \sim 0.7$  V/m at  $T = 25$  mK. So  $\mathcal{E}/\mathcal{E}_c \sim 5 \times 10^3$ , and  $J/J_c = (\mathcal{E}/\mathcal{E}_c)^2 \sim 2 \times 10^7 \gg 1$ . We can do a similar estimate to show that a single photon would even more strongly saturate resonant TLS in the insulating barrier of the tunnel junction. We use the same numbers as before but with  $C = 1$  fF and the thickness of the junction  $L = 1.5$  nm to obtain  $\mathcal{E} \sim 7 \times 10^4$  V, and  $J/J_c \sim 10^{10} \gg 1$ . However, there are only a few TLS in the oxide barrier of a small tunnel junction. For a parallel plate capacitor with  $L = 1.5$  nm and  $A = 1$   $\mu$ m<sup>2</sup>, the volume is  $V_o = 1.5 \times 10^{-21}$  m<sup>3</sup>. With a density of states  $P_{TLS} \simeq 10^{45}/(Jm^3) \simeq 663/h\text{GHz}\mu\text{m}^3$ , there are only 2 TLS with an energy splitting less than 10 GHz. A single fluctuator would have a Lorentzian noise spectrum. The presence of  $1/f$  noise implies many more than 2 fluctuators. It is likely that these additional fluctuators are in the substrate. Our main point is that TLS in small devices are easily saturated. It is therefore important to analyze the effect of TLS saturation on the charge noise both at low and high frequencies  $f$  of the noise spectrum.

In this article, we explore the consequences of this saturation on the spectral density of polarization and charge fluctuations. We consider a driven system consisting of two level systems with electric dipole moments that fluctuate randomly, leading to fluctuations  $\delta P(t)$  in the polarization. In addition, the dipole moments of these TLS couple to an applied ac electric field that drives the system with an angular frequency  $\Omega$ .

Let us consider a single two level system. Let the density matrix  $\rho$  of this two level system contain the contributions of the driving field but not the random fluctuations of the dipoles. According to the Wiener-Khinchine theorem, in the stationary state the polarization noise spectral density  $S_P(\omega)$  of this two level system is twice the Fourier transform of the polarization autocorrelation

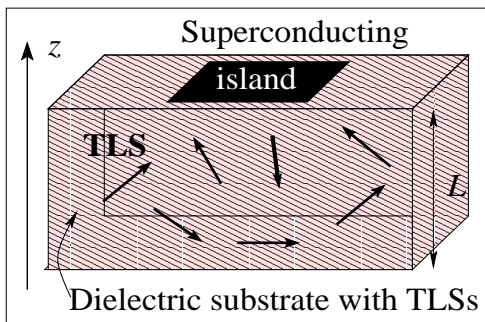


FIG. 1: Distribution of two level systems (TLS) in the dielectric substrate of a Cooper pair box. The superconducting island with the Josephson junction qubit is the black rectangle on the top of the substrate.

function

$$\begin{aligned} S_P(\omega) &= 2 \int_{-\infty}^{+\infty} d(t_1 - t_2) e^{i\omega(t_1 - t_2)} [\langle \delta P(t_1) \delta P(t_2) \rangle]_\rho \\ &= 2 \int_{-\infty}^{+\infty} d(t_1 - t_2) e^{i\omega(t_1 - t_2)} \text{Tr}\{\rho \langle \delta P(t_1) \delta P(t_2) \rangle\} \end{aligned}$$

where  $[\dots]_\rho$  denotes a trace over the density matrix  $\rho$ , and  $\langle \dots \rangle$  denotes an average over the time series. The density matrix contains the contributions of the driving field to  $S_P(\omega)$  while the polarization autocorrelation function  $\langle \delta P(t_1) \delta P(t_2) \rangle$  contains the contributions of the random fluctuations of the electric dipoles. We will solve the Bloch-Redfield equations in steady state to find the time evolution of the density matrix  $\rho(t)$  and its dependence on the ratio  $J/J_c(\Omega, T)$  of the electromagnetic flux  $J$  to the critical flux  $J_c(\Omega, T)$  which is a function of the driving angular frequency  $\Omega$  and temperature  $T$ . The benefit of this approach is that it is valid in both equilibrium as well as steady state non-equilibrium situations. From the polarization noise spectral density, we can obtain the charge noise  $S_Q(\omega)$ . We then average over the distribution of independent TLS. Unlike previous theoretical efforts,<sup>8–10</sup> we use the standard TLS density of states that is a constant independent of energy.

At low frequencies ( $hf \ll k_B T$ ) the system is in equilibrium, and we find  $1/f$  charge noise that is proportional to the temperature and to the dielectric loss tangent  $\tan \delta$  which has a well known contribution from the electric dipole moments of TLS.<sup>11,16,17</sup> In addition the low frequency charge noise has negligible dependence on the incident electromagnetic flux ratio  $J/J_c$ . At high frequencies ( $hf \gg k_B T$ ) we find that the charge noise is white noise independent of frequency with a small value of approximately  $4 \times 10^{-17} \text{ Hz}^{-1}$ . It has a very weak dependence on the ratio  $J/J_c(\Omega, T)$  and the driving frequency  $\Omega$ . We also find that the amplitude of the high frequency white charge noise decreases gradually as the temperature increases. The fact that the charge noise spectrum depends very weakly on the ratio  $J/J_c(\Omega, T)$  indicates that the saturation of two level systems does not affect charge noise.

The paper is organized as follows. In Section II, we present our model of a TLS in an external driving field. In Section III, we use the fluctuation-dissipation theorem to give an expression for the charge noise in thermal equilibrium. In Section IV, we take a more general approach that is valid in both equilibrium and nonequilibrium cases. In particular, we derive a general analytic expression for the spectral density of polarization and charge fluctuations of an individual two level system (also referred to as a fluctuator) in terms of the density matrix. In Section V, we solve the Bloch-Redfield linear differential equations for the density matrix. We find the steady state solution of the Bloch-Redfield equations and we analyze its dependence on the ratio  $J/J_c(\Omega, T)$ . In Section VI, we investigate the noise spectrum of a single random telegraph fluctuator. We then average over the distribution of independent TLS numerically to determine the frequency dependence of the noise spectrum. A summary is given in Section VII.

## II. TWO LEVEL SYSTEM (TLS)

In applying the standard model of two level systems to Josephson junction devices, we consider a TLS that sits in the insulating substrate or in the tunnel barrier, and has an electric dipole moment  $\mathbf{p}$  consisting of a pair of opposite charges separated by a distance  $d$ . The electrodes positioned at  $z = 0$  and  $z = L$  are kept at the same potential. The angle between  $\mathbf{p}$  and the  $z$ -axis, perpendicular to the electrodes, is  $\theta$ . The dipole flips and induces polarization fluctuations and hence charge fluctuations on the electrodes. These induced charges  $Q$  are proportional to the  $z$ -component of the dipole moment,  $p_z = p \cos \theta$ , i.e.,  $Q = |p \cos \theta|/L$ .

The TLS is in a double-well potential with a tunneling matrix element  $\Delta_0$  and an asymmetry energy  $\Delta$ .<sup>16</sup> The Hamiltonian of a TLS in an external ac field can be written as  $H(t) = H_0 + H_1(t)$ , where  $H_0 = \frac{1}{2}(\Delta\sigma_z + \Delta_0\sigma_x)$ , and  $H_1(t) = -\sigma_z \mathbf{p} \cdot \boldsymbol{\xi}_{ac}(t)$ . Here  $\sigma_{x,z}$  are the Pauli spin matrices and  $\boldsymbol{\xi}_{ac}(t) = \boldsymbol{\xi}_{ac} \cos \Omega t$  is a small perturbing ac electric field of angular frequency  $\Omega$  that couples to the TLS electric dipole moment.  $\boldsymbol{\xi}_{ac}$  points along the  $z$ -axis.

After diagonalization of  $H_0$ , the Hamiltonian becomes

$$H(t) = H_0 + H_1(t) \quad (1)$$

$$H_0 = \frac{1}{2} E \sigma_z \quad (2)$$

$$H_1(t) = -\eta(\Delta_0\sigma_x + \Delta\sigma_z)\cos\Omega t, \quad (3)$$

where  $E = \sqrt{\Delta^2 + \Delta_0^2}$  is the TLS energy splitting and  $\eta \equiv \mathbf{p} \cdot \boldsymbol{\xi}_{ac}/E$ . Notice that  $\eta$  is a small dimensionless variable ( $\eta \approx 5 \times 10^{-3}$  for  $p = 3.7 \text{ D}$ ,  $\xi_{ac} \approx 10^3 \text{ V/m}$  and  $E/h \approx 10 \text{ GHz}$ ). The energy eigenbasis is denoted by  $\{|+\rangle, |-\rangle\}$ , and the corresponding eigenvalues are  $E_\pm = \pm E/2$ , where  $+$  ( $-$ ) refers to the upper (lower) level of the TLS. The energy splitting  $E$  will also be referred to as  $\hbar\omega_0$ .

An excited two level system can decay to the ground state by emitting a phonon. The longitudinal relaxation rate is given by:<sup>16</sup>

$$T_1^{-1} = \frac{\gamma_d^2}{(2\pi\rho\hbar^4)} \left[ \left( \frac{1}{c_\ell^5} \right) + \left( \frac{2}{c_t^5} \right) \right] E\Delta_o^2 \coth(E/2k_B T), \quad (4)$$

where  $\rho$  is the mass density,  $c_\ell$  is the longitudinal speed of sound,  $c_t$  is the transverse speed of sound, and  $\gamma_d$  is the deformation potential. In this paper we will use the values for SiO<sub>2</sub>:  $\gamma_d = 1$  eV,  $\rho = 2200$  kg/m<sup>3</sup>,  $c_\ell = 5800$  m/s, and  $c_t = 3800$  m/s. Typically,  $T_1$  varies between  $T_{1,min} = 10^{-9}$  s and  $T_{1,max} = 10^4$  s for temperatures around 0.1 K. The distribution of TLS parameters is given by<sup>16,18</sup>

$$P(E, T_1) = \frac{P_{TLS}}{2T_1 \sqrt{1 - (\tau_{min}(E)/T_1)}}, \quad (5)$$

where  $P_{TLS}$  is a constant density of states that represents the number of TLS per unit energy and unit volume. The minimum relaxation time  $\tau_{min}(E)$  corresponds to a symmetric double-well potential (i.e.,  $E = \Delta_0$ ). Alternatively, the TLS distribution function can be expressed in terms of the TLS matrix elements  $\Delta$  and  $\Delta_0$ :

$$P(\Delta, \Delta_0) = \frac{P_{TLS}}{\Delta_0}. \quad (6)$$

The typical range of values for  $\Delta$  and  $\Delta_0$  are  $0 \leq \Delta/k_B \leq 4$  K and  $2 \mu\text{K} \leq \Delta_0/k_B \leq 4$  K, where  $k_B$  is the Boltzmann's constant. We will use these values for our numerical integrations in Section VI.

### III. THERMAL EQUILIBRIUM EXPRESSION FOR CHARGE NOISE

We begin by considering the case of thermal equilibrium. According to the Wiener-Khintchine theorem, the charge spectral density  $S_Q(\omega)$  is twice the Fourier transform  $\Psi_Q(\omega)$  of the autocorrelation function of the fluctuations in the charge. In equilibrium we can use the fluctuation-dissipation theorem<sup>19</sup> to find that the (un-symmetrized) charge noise is given by:

$$S_Q(\mathbf{k}, \omega) = \frac{4\hbar}{1 - e^{-\hbar\omega/k_B T}} \chi_Q''(\mathbf{k}, \omega), \quad (7)$$

where  $Q$  is the induced (bound) charge on the electrodes and  $\chi_Q''(\mathbf{k}, \omega)$  is the Fourier transform of

$$\chi_Q''(\mathbf{r}, t; \mathbf{r}', t') = \frac{\langle [Q(\mathbf{r}, t), Q(\mathbf{r}', t')]_c \rangle_e}{2\hbar} \quad (8)$$

where  $[...]_c$  is a commutator, and  $\langle \dots \rangle_e$  is an ensemble average. We use  $Q = \int \mathbf{P} \cdot d\mathbf{A}$ , where  $\mathbf{P}$  is the electric polarization density, and choose  $P_z$  and  $d\mathbf{A} \parallel \hat{z}$  since  $Q \sim |p_z|$  to find

$$\chi_Q''(\mathbf{k}, \omega) = \varepsilon_o A^2 \chi_{P_z}''(\mathbf{k}, \omega) \quad (9)$$

where  $\varepsilon_o$  is the vacuum permittivity,  $A$  is the area of a plate of the parallel plate capacitor with capacitance  $C$ , and  $\chi_{P_z}''(\mathbf{k}, \omega)$  is the imaginary part of the electric susceptibility. We set  $\mathbf{k} = 0$ , and use

$$\varepsilon_o \chi_{P_z}''(\omega) = \varepsilon'(\omega) \tan \delta(\omega) \quad (10)$$

where  $\tan \delta(\omega)$  is the dielectric loss tangent, and  $\varepsilon'(\omega)$  is the real part of the dielectric permittivity. We also use

$$C = \varepsilon' A/L \quad (11)$$

to find

$$S_Q(\omega) = \frac{4\hbar C}{1 - e^{-\hbar\omega/k_B T}} \tan \delta(\omega), \quad (12)$$

where  $S_Q(\omega) \equiv S_Q(\mathbf{k} = 0, \omega)/V_o$ , the volume of the capacitor is  $V_o = AL$ , and  $\varepsilon'(\omega) = \varepsilon' + \varepsilon_{TLS}(\omega) \simeq \varepsilon' = \varepsilon_o \varepsilon_r$  where  $\varepsilon_r$  is the relative permittivity. The frequency dependent permittivity  $\varepsilon_{TLS}(\omega)$  produced by TLS is negligible compared to the constant permittivity  $\varepsilon'$ .<sup>16</sup> The TLS dynamic electric susceptibilities ( $\chi'(\omega)$ ,  $\chi''(\omega)$ ), and hence the dielectric loss tangent, can be obtained by solving the Bloch equations in equilibrium.<sup>11,13,20</sup> One can then average over the distribution of TLS parameters. However, since we will be considering driven systems that are in a nonequilibrium steady state, we need to take the more general approach that is described in the next section.

### IV. GENERAL EXPRESSION FOR SPECTRAL DENSITY OF POLARIZATION AND CHARGE FLUCTUATIONS OF A TWO LEVEL SYSTEM

The noise in our model is due to a fluctuating two level system with an electric dipole moment that changes its orientation with respect to the direction of the applied driving field while keeping its magnitude constant. In this section, we begin by deriving a general expression for the polarization noise  $S_P(\omega)$  of a single TLS that is valid at all frequencies and in both equilibrium and nonequilibrium situations. Since we are interested in TLS saturation, this formulation will apply to a driven system in nonequilibrium steady state. We then relate the polarization noise to the charge noise  $S_Q(\omega)$ .

According to the Wiener-Khintchine theorem, the polarization spectral density  $S_P(\omega)$  in the stationary state is twice the Fourier transform of the autocorrelation function of the fluctuations in the polarization:

$$S_P(\omega) = 2 \int_{-\infty}^{+\infty} d(t_1 - t_2) e^{i\omega(t_1 - t_2)} [\langle \delta P_H(t_1) \delta P_H(t_2) \rangle]_\rho. \quad (13)$$

The subscript  $H$  denotes the Heisenberg representation. We can rewrite this expression in the Heisenberg representation as

$$S_P(\omega) = 2 \int_{-\infty}^{+\infty} d(t_1 - t_2) e^{i\omega(t_1 - t_2)} \times \text{Tr}\{\rho_H \langle \delta P_H(t_1) \delta P_H(t_2) \rangle\}, \quad (14)$$

where the density matrix  $\rho_H$  is time independent in the Heisenberg representation, but not in the Schrodinger representation. We now change from the Heisenberg representation to the Schrodinger representation (denoted by the subscript  $S$ ). Recall that an operator  $A_H(t)$  in the Heisenberg representation can be expressed in terms of its version in the Schrodinger representation by  $A_H(t) = U^+(t, t_0)A_S(t)U(t, t_0)$ , where  $U(t, t_0)$  is the unitary evolution operator. Hence,

$$\text{Tr}\{\rho_H\delta P_H(t_1)\delta P_H(t_2)\} = \text{Tr}\{\rho_S(t_1)\delta P_S(t_1)U(t_1, t_2)\delta P_S(t_2)U(t_2, t_1)\} \quad (15)$$

$$\equiv F(t_1, t_2). \quad (16)$$

To simplify the notation we have temporarily omitted the symbol  $\langle \dots \rangle$  denoting the time average. The spectral density of polarization fluctuations in the Schrodinger representation becomes:

$$S_P(\omega) = 2 \int_{-\infty}^{+\infty} d(t_1 - t_2) e^{i\omega(t_1 - t_2)} F(t_1, t_2), \quad (17)$$

where the function  $F(t_1, t_2)$  depends on  $(t_1 - t_2)$  for stationary processes. It can be expressed as

$$\begin{aligned} F(t_1, t_2) &= \sum_{m, n, p} \langle m(t_1) | \rho(t_1) | n(t_1) \rangle \langle n(t_1) | \delta P(t_1) | p(t_1) \rangle \\ &\quad \times \langle p(t_1) | U(t_1, t_2) \delta P(t_2) U(t_2, t_1) | m(t_1) \rangle \\ &= \sum_{m, n, p} \langle m(t_1) | \rho(t_1) | n(t_1) \rangle \langle n(t_1) | \delta P(t_1) | p(t_1) \rangle \\ &\quad \times \langle p(t_2) | \delta P(t_2) | m(t_2) \rangle, \end{aligned} \quad (18)$$

where  $\rho(t)$  is the density matrix in the Schrodinger representation, and  $m$ ,  $n$ , and  $p$  denote eigenstates of  $H_0$ . As we mentioned in the introduction, we are considering the density matrix  $\rho$  of a single TLS that contains the time dependence of the external driving field. The random dipole fluctuations are contained in  $\delta P_H(t)$ . Let  $\alpha$  stand for  $m$ ,  $n$ , or  $p$ . Then  $|\alpha(t)\rangle = \exp(-iE_\alpha t/\hbar)|\alpha\rangle$  and  $H_0|\alpha\rangle = E_\alpha|\alpha\rangle$ . We now switch from  $|n(t)\rangle$ ,  $|m(t)\rangle$ , and  $|p(t)\rangle$  to the  $|+\rangle$  and  $|-\rangle$  eigenstates of a TLS to obtain:

$$\begin{aligned} F(t_1, t_2) &= [\rho(t_1)\delta P(t_1)]_{++}\delta P_{++}(t_2) \\ &\quad + [\rho(t_1)\delta P(t_1)]_{--}\delta P_{--}(t_2) \\ &\quad + e^{-i\omega_0(t_1 - t_2)}[\rho(t_1)\delta P(t_1)]_{-+}\delta P_{+-}(t_2) \\ &\quad + e^{+i\omega_0(t_1 - t_2)}[\rho(t_1)\delta P(t_1)]_{+-}\delta P_{-+}(t_2), \end{aligned} \quad (19)$$

where  $\delta P_{\alpha\alpha'}(t)$  denotes the  $\alpha\alpha'$ th element of the  $\delta P(t)$  matrix,  $[\rho(t)\delta P(t)]_{\alpha\alpha'}$  represents the  $\alpha\alpha'$ th element of the  $\rho(t)\delta P(t)$  matrix, and  $\omega_0 \equiv E/\hbar$ . We will see in Section V that  $\rho_{+-}$  and  $\rho_{-+}$  are first order in the small parameter  $\eta = \mathbf{p} \cdot \boldsymbol{\xi}_{ac}/E$ .  $\eta \ll 1$  for both small and large values of  $J/J_c$ . We assume that the fluctuations in the polarization  $\delta P$  are also small and of order  $\eta$ . Hence the terms involving products such as  $\rho_{-+}\delta P_{+-}\delta P_{--}$ ,  $\rho_{+-}\delta P_{-+}\delta P_{++}$ ,  $\rho_{-+}\delta P_{++}\delta P_{+-}$ , and  $\rho_{+-}\delta P_{--}\delta P_{-+}$  are of third order

in small perturbations and can be neglected. This leads to an approximate expression for  $F(t_1, t_2)$ :

$$\begin{aligned} F(t_1, t_2) &\approx \rho_{++}\delta P_{++}(t_1)\delta P_{++}(t_2) \\ &\quad + \rho_{--}\delta P_{--}(t_1)\delta P_{--}(t_2) \\ &\quad + e^{-i\omega_0(t_1 - t_2)}\rho_{--}\delta P_{-+}(t_1)\delta P_{+-}(t_2) \\ &\quad + e^{+i\omega_0(t_1 - t_2)}\rho_{++}\delta P_{-+}(t_1)\delta P_{-+}(t_2). \end{aligned} \quad (20)$$

Let  $P_{||}(t)$  be the projection along the ac external field of the polarization operator associated with the dipole moment  $p$  of a two level system.  $P_{||}(t)$  has stochastic fluctuations due to the fact that the electric dipole moment of the two level system randomly changes its orientation angle  $\theta(t)$  with respect to the applied electric field. Hence, we can write

$$\begin{aligned} P_{||}(t) &= -\frac{p \cos(\theta(t))}{V_o} \left( \frac{\Delta_0}{E} \sigma_x + \frac{\Delta}{E} \sigma_z \right) \\ &\equiv P_0(t) \left( \frac{\Delta_0}{E} \sigma_x + \frac{\Delta}{E} \sigma_z \right), \end{aligned} \quad (21)$$

where  $P_0(t) \equiv -p \cos(\theta(t))/V_o$  and  $V_o$  is volume. Using Eqs. (17) and (20), we obtain

$$\begin{aligned} S_{P_{||}}(\omega) &= 2 \int_{-\infty}^{+\infty} d(t_1 - t_2) e^{i\omega(t_1 - t_2)} \langle P_0(t_1) P_0(t_2) \rangle \\ &\quad \times \left\{ \left( \frac{\Delta}{E} \right)^2 [1 - \langle P_{||}^2 \rangle] + \left( \frac{\Delta_0}{E} \right)^2 [e^{-i\omega_0(t_1 - t_2)} \rho_{--} \right. \\ &\quad \left. + e^{i\omega_0(t_1 - t_2)} \rho_{++}] \right\}, \end{aligned} \quad (22)$$

where  $\langle P_{||} \rangle$  is the time averaged value of the operator  $P_{||}(t)$ . Since for stationary processes the correlator  $\langle P_0(t_1) P_0(t_2) \rangle$  is a function of  $(t_1 - t_2)$ , we can define  $S_{P_0}(\omega) \equiv 2 \int_{-\infty}^{+\infty} d(t_1 - t_2) e^{i\omega(t_1 - t_2)} \langle P_0(t_1) P_0(t_2) \rangle$ . Then we have

$$\begin{aligned} S_{P_{||}}(\omega) &\equiv 2 \int_{-\infty}^{+\infty} d(t_1 - t_2) e^{i\omega(t_1 - t_2)} \\ &\quad \times \langle P_0(t_1) P_0(t_2) \rangle \\ &= \left( \frac{\Delta}{E} \right)^2 [1 - \langle P_{||}^2 \rangle] S_{P_0}(\omega) \\ &\quad + \left( \frac{\Delta_0}{E} \right)^2 [\rho_{--} S_{P_0}(\omega - \omega_0) \\ &\quad + \rho_{++} S_{P_0}(\omega + \omega_0)]. \end{aligned} \quad (23)$$

This is a general formula for the spectral density of the polarization fluctuations assuming that the fluctuations in the orientations of the electric dipole moments of TLS are a stationary process. It has two contributions. The first term, which is proportional to  $S_{P_0}(\omega)$ , is the *relaxation* (REL) contribution. It is associated with the TLS pseudospin  $\sigma_z$  whose expectation value is proportional to the population difference between the two levels of the TLS. The relaxation contribution to phonon

or photon attenuation is due to the modulation of the TLS energy splitting  $E$  by the incident photons which have energy  $\hbar\omega \ll E$ . This modulation causes the population of the TLS energy levels to readjust which requires energy and leads to attenuation of the incident flux. The last two terms in Eq. (24) are proportional to  $S_{P_0}(\omega \pm \omega_0)$  and are *resonance* (RES) contributions. The resonance terms are associated with the  $x$  and  $y$  components of the TLS pseudospin that describe transitions between levels. They describe the resonant absorption by TLS of photons or phonons with  $\hbar\omega = E$ . We will see in section VI that the relaxation term is dominant at low frequencies, while the resonance contributions are dominant at high frequencies.

To obtain the charge noise  $S_Q(\omega)$  from the polarization noise, we make use of the following formulas. In a polarized medium, the induced (bound) charge is

$$Q = \int \mathbf{P} \cdot d\mathbf{A} \quad (25)$$

where  $\mathbf{P}$  is the electric polarization. We choose  $P_z$  and  $d\mathbf{A} \parallel \hat{z}$  since  $Q \sim |p_z| = |p \cos \theta|$ . Then

$$S_Q(\omega) = A^2 S_{P_z}(\omega) \quad (26)$$

$$= \frac{A^2 S_{P_{||}}(\omega)}{3}, \quad (27)$$

where  $A$  is the area of a plate of a parallel plate capacitor with capacitance  $C = \epsilon' A/L$ , and  $\epsilon'$  is the real part of the dielectric permittivity.

In this section we have derived expressions for the polarization and charge noise for a single TLS in terms of the density matrix. In the next section we will solve the Bloch-Redfield equations for the time dependent density matrix of a TLS subjected to an external ac driving field.

## V. THE BLOCH-REDFIELD EQUATIONS

From Eqs. (17) and (18), we see that we need the time dependent density matrix to calculate the polarization noise spectrum. In this section, we solve the Bloch-Redfield equations to find the time evolution of the density matrix of a single TLS subject to an external ac electric field. These equations combine the equation of motion of the density matrix with time-dependent perturbation theory, taking into account the relaxation and dephasing of TLS.

We follow Slichter<sup>21</sup> and write the following set of linear differential equations for the density matrix elements  $\rho_{\alpha\alpha'}(t)$ :

$$\begin{aligned} \frac{d\rho_{\alpha\alpha'}}{dt} &= \frac{i}{\hbar} \langle \alpha | [\rho, H_0] | \alpha' \rangle \\ &+ \frac{i}{\hbar} \langle \alpha | [\rho, H_1(t)] | \alpha' \rangle \\ &+ \sum_{\beta, \beta'} R_{\alpha\alpha', \beta\beta'} [\rho_{\beta\beta'} - \rho_{\beta\beta'}(T)], \end{aligned} \quad (28)$$

where  $\alpha$  and  $\beta$  can be either  $+$  or  $-$ , corresponding to the energy eigenstates of the TLS, and  $R_{\alpha\alpha', \beta\beta'}$  are the Bloch-Redfield tensor components which are constant in time. They are related to the longitudinal and transverse relaxation times,  $T_1$  and  $T_2$ . In Eq. (28),  $H_0$  and  $H_1(t)$  are Hamiltonians given by Eqs. (2) and (3), respectively. The thermal equilibrium value of the density matrix is denoted by  $\rho^{eq}(T)$ . At thermal equilibrium, only the diagonal elements of the density matrix are nonzero, and are given by  $\rho_{--}^{eq}(T) = \exp(+E/2)/Z$  and  $\rho_{++}^{eq}(T) = \exp(-E/2)/Z$ , where the partition function  $Z = [\exp(-E/2) + \exp(+E/2)]$ . Eq. (28) can be written in the form:

$$\begin{aligned} \frac{d\rho_{\alpha\alpha'}}{dt} &= \frac{i}{\hbar} (E_{\alpha'} - E_{\alpha}) \rho_{\alpha\alpha'} + \frac{i}{\hbar} \sum_{\alpha''} [\rho_{\alpha\alpha''} \langle \alpha'' | H_1(t) | \alpha' \rangle \\ &- \langle \alpha | H_1(t) | \alpha'' \rangle \rho_{\alpha''\alpha'}] \\ &+ \sum_{\beta, \beta'} R_{\alpha\alpha', \beta\beta'} [\rho_{\beta\beta'} - \rho_{\beta\beta'}^{eq}(T)]. \end{aligned} \quad (29)$$

Next we use the fact that in the relaxation terms  $R_{\alpha\alpha', \beta\beta'}$ , the only important terms correspond to<sup>21,22</sup>  $\alpha - \alpha' = \beta - \beta'$ . In addition, the Bloch-Redfield tensor is symmetric, so we have the following relations for the dominant components:

$$R_{--,++} = R_{+,-,-} \equiv \frac{1}{T_1} \quad (30)$$

$$R_{-+,-+} = R_{+-,+ -} \equiv -\frac{1}{T_2}. \quad (31)$$

The longitudinal relaxation time  $T_1$  is given by Eq. (4). For the transverse relaxation time  $T_2$ , we will use the experimental value<sup>23</sup>

$$T_2 \approx \frac{8 \times 10^{-7}}{T} \text{ sec}, \quad (32)$$

where  $T$  is in Kelvin. From relations (30) and (31), the set of linear differential equations (29) becomes

$$\begin{aligned} \frac{d\rho_{--}}{dt} &= \frac{i}{\hbar} (\rho_{-+} [H_1(t)]_{+-} - [H_1(t)]_{-+} \rho_{-+}) \\ &+ \frac{1}{T_1} (\rho_{++} - \rho_{++}^{eq}(T)), \end{aligned} \quad (33)$$

$$\begin{aligned} \frac{d\rho_{++}}{dt} &= \frac{i}{\hbar} (\rho_{+-} [H_1(t)]_{-+} - [H_1(t)]_{+-} \rho_{+-}) \\ &+ \frac{1}{T_1} (\rho_{--} - \rho_{--}^{eq}(T)), \end{aligned} \quad (34)$$

$$\begin{aligned} \frac{d\rho_{-+}}{dt} &= \frac{i}{\hbar} (E_+ - E_-) \rho_{-+} \\ &+ \frac{i}{\hbar} (\rho_{--} [H_1(t)]_{-+} - [H_1(t)]_{--} \rho_{-+} \\ &+ \rho_{-+} [H_1(t)]_{++} - [H_1(t)]_{-+} \rho_{++}) - \frac{1}{T_2} \rho_{-+}, \end{aligned} \quad (35)$$

$$\frac{d\rho_{+-}}{dt} = \frac{d\rho_{-+}^*}{dt}, \quad (36)$$

where  $[H_1(t)]_{\alpha,\beta} = \langle \alpha | H_1(t) | \beta \rangle$ . Using Eq. (3), we can write the first two equations for the diagonal elements as:

$$\begin{aligned} \frac{d\rho_{--}}{dt} &= -\frac{d\rho_{++}}{dt} = \frac{i}{\hbar}(-\eta\Delta_0\cos\Omega t)(\rho_{-+} - \rho_{+-}) \\ &+ \frac{1}{T_1}[\rho_{++} - \rho_{--} - (\rho_{++}^{eq}(T) - \rho_{--}^{eq}(T))], \end{aligned} \quad (37)$$

while the equation for the off-diagonal element  $\rho_{-+}$  is:

$$\begin{aligned} \frac{d\rho_{-+}}{dt} &= \frac{i}{\hbar}(E_+ - E_-)\rho_{-+} + \frac{i}{\hbar}(\rho_{--} - \rho_{++})(-\eta\Delta_0\cos\Omega t) \\ &+ \frac{i}{\hbar}\rho_{-+}(-2\eta\Delta_0\cos\Omega t) - \frac{1}{T_2}\rho_{-+}. \end{aligned} \quad (38)$$

We look for a steady state solution of the form

$$\rho_{--} = r_{--}, \quad \rho_{++} = r_{++} \quad (39)$$

$$\rho_{-+}(t) = r_{-+}e^{i\Omega t}, \quad \rho_{+-}(t) = \rho_{-+}^*(t), \quad (40)$$

where  $r_{\alpha\alpha'}$  are complex constants. We find that in steady state the density matrix elements are given by the following expressions:

$$\rho_{--} = \frac{1}{2} + \frac{\rho_{--}^{eq}(T) - 1/2}{1 + g(\Omega, \omega_0, T_2) \times (J/J_c)} \quad (41)$$

$$\rho_{++} = \frac{1}{2} + \frac{\rho_{++}^{eq}(T) - 1/2}{1 + g(\Omega, \omega_0, T_2) \times (J/J_c)} \quad (42)$$

$$\begin{aligned} \rho_{-+}(t) &= -\frac{iT_2\eta\Delta_0}{2\hbar} \frac{1}{[1 + iT_2(\Omega - \omega_0)]} \\ &\times \frac{\rho_{--}^{eq}(T) - \rho_{++}^{eq}(T)}{1 + g(\Omega, \omega_0, T_2) \times (J/J_c)} \exp(i\Omega t) \end{aligned} \quad (43)$$

$$\rho_{+-}(t) = \rho_{-+}^*(t), \quad (44)$$

where  $\eta = \mathbf{p} \cdot \boldsymbol{\xi}_{ac}/E$ ,  $J/J_c = T_1T_2(\eta\Delta_0/\hbar)^2/2$ , and  $g(\Omega, \omega_0, T_2) = 1/[1 + (\Omega - \omega_0)^2T_2^2]$ . For a dipole moment  $p = 3.7$  D, a large electric field  $\xi_{ac} = 3 \times 10^3$  V/m, and TLS energy splittings of the order of 10 GHz, the dimensionless factor  $\eta \approx 0.005$ , and it decreases to a value of  $5 \times 10^{-8}$  when the amplitude of the applied electric field is  $\xi_{ac} = 0.03$  V/m.  $g(\Omega, \omega_0, T_2)$  is approximately equal to 1 when the ac driving frequency is resonant with the TLS energy splitting, i.e.,  $\Omega \approx \omega_0$ .

Notice that the off-diagonal elements of the density matrix are first order in  $\eta \ll 1$ . They are oscillatory and small, as shown by the following numerical estimate. For large electric fields ( $\xi_{ac} = 3 \times 10^3$ ),  $\eta \approx 0.005$ ,  $T_2 = 8$   $\mu$ s at  $T = 0.1$  K,<sup>23,24</sup>  $\Delta_0/\hbar \approx E/\hbar \approx 10$  GHz, and  $T_1 = 8 \times 10^{-8}$  s, we obtain  $J/J_c = T_1T_2(\eta\Delta_0/\hbar)^2/2 \approx 10^7 \gg 1$ , and  $|\rho_{-+}| \approx 10^{-4}$ . For  $J \ll J_c$ , we have  $|\rho_{-+}| \approx 10^{-2}$ . Hence, the amplitude of the off-diagonal density matrix elements is very small. On the other hand, the diagonal elements have small deviations from their equilibrium values (at low electromagnetic fields  $J \ll J_c$ ) or from their steady state values (at large electromagnetic fields  $J \gg J_c$ ). Also, as required,  $\text{Tr}\{\rho\} = \rho_{--} + \rho_{++} = 1$ . In addition, the quantum expectation value of the

$z$ -component of the TLS spin is

$$\langle S_z \rangle = (\rho_{++} - \rho_{--}) \quad (45)$$

$$= -\frac{\tanh(E/(2k_B T))}{[1 + (J/J_c) \times g(\Omega, \omega_0, T_2)]}. \quad (46)$$

Note that all the elements of the density matrix depend on the ratio  $J/J_c(\Omega, T)$ . To approximate the populations of the upper and lower TLS energy levels for small and large electromagnetic fields, we write down approximate expressions of  $\rho_{--}$  and  $\rho_{++}$ . For  $J \ll J_c$  and  $g \approx 1$ , we expand  $\rho_{--}$  and  $\rho_{++}$  to first order in  $J/J_c$  to obtain:

$$\rho_{--} \approx \rho_{--}^{eq}(T)(1 - \frac{J}{J_c}g) + \frac{1}{2}\frac{J}{J_c}g \quad (47)$$

$$\rho_{++} \approx \rho_{++}^{eq}(T)(1 - \frac{J}{J_c}g) + \frac{1}{2}\frac{J}{J_c}g. \quad (48)$$

This means that *unsaturated* TLS with energy splittings of the order of 10 GHz at a temperature of 0.1 K have  $\rho_{++} \approx 0$  and  $\rho_{--} \approx 1$ . Thus the upper level is mostly unoccupied while the lower level is almost always occupied.

On the other hand, for  $J \gg J_c$ , we can expand the steady state solution for  $\rho_{--}$  and  $\rho_{++}$  given in Eqs. (41) and (42) to first order in  $(J/J_c)^{-1}$ . The result is

$$\rho_{--} \approx \frac{1}{2} + [\rho_{--}^{eq}(T) - \frac{1}{2}](\frac{J}{J_c}g)^{-1} \quad (49)$$

$$\rho_{++} \approx \frac{1}{2} - [\rho_{--}^{eq}(T) - \frac{1}{2}](\frac{J}{J_c}g)^{-1}. \quad (50)$$

Hence, for *saturated* TLS the lower and upper levels are almost equally populated, i.e.,  $\rho_{++} \approx 1/2$  and  $\rho_{--} \approx 1/2$ . This is to be expected since the TLS are constantly being excited by the ac electric field and de-excited by spontaneous and stimulated emission. Once the TLS are saturated, the populations of the lower and upper levels will have small deviations from their steady state values. We will look at the saturation effect in more detail in Section VI where we plot noise spectrum of a single fluctuator versus  $J/J_c$ . This noise spectrum depends on the density matrix elements  $\rho_{++}$  and  $\rho_{--}$ . As one goes from the unsaturated regime to the saturated regime, the amplitude of  $\rho_{--}$  decreases by a factor of two. From Eq. (24) the polarization noise of a single TLS depends linearly on  $\rho_{--}$ . Because  $\rho_{--}$  only decreases by a factor of 2 when the TLS are saturated, we will see that the saturation of TLS will not play an important role in the polarization and charge noise spectra.

## VI. RESULTS

In this section we begin by studying the polarization noise spectrum of a single TLS fluctuator as a function of frequency and electromagnetic flux ratio ( $J/J_c$ ). We then obtain the total polarization noise by averaging over the distribution of independent two level systems. From this

we get the charge noise which we will analyze at both low and high frequencies as a function of temperature and incident electromagnetic flux.

### A. Polarization Noise of One TLS Fluctuator

Now that we have the matrix elements of the steady state density matrix in Eqs. (41)–(44), we can use them to evaluate the expression for polarization noise found in Eq. (24). In order to make further progress in evaluating Eq. (24), we need to know the polarization noise spectrum  $S_{P_0}(\omega)$  of a TLS. So we assume that a single TLS fluctuates randomly in time. Its electric dipole moment fluctuates in orientation by making  $180^\circ$  flips between  $\theta = \theta_0$  and  $(\theta = \theta_0 + 180^\circ)$ , resulting in a random telegraph signal (RTS) in the polarization  $P_0(t) = \pm p \cos(\theta_0)/V$  along the external field. The noise spectrum is a Lorentzian given by<sup>25</sup>

$$S_{P_0}(\omega) = \frac{\langle \delta P_0^2 \rangle \tau}{(1 + \omega^2 \tau^2)}, \quad (51)$$

where  $\tau$  is the characteristic relaxation time of the fluctuator. So in Eq. (24), we replace  $S_{P_0}(\omega)$  by the RTS noise spectrum with a  $T_1$  relaxation time, and  $S_{P_0}(\omega \pm \omega_0)$  by the corresponding RTS noise spectrum with a  $T_2$  relaxation time.

In evaluating Eq. (24) we also need  $\langle P_{||} \rangle$  which enters into from  $\delta P_{||} = P_{||} - \langle P_{||} \rangle$ . We do not know what  $\langle P_{||} \rangle$  is when the system is in a nonequilibrium driven steady state, but we do know it in thermal equilibrium. So we set  $\langle P_{||} \rangle$  equal to its thermal equilibrium value, i.e.,  $\langle P_{||} \rangle = -\tanh(E/2k_B T)$ . If we did not subtract the mean polarization, then the noise spectrum associated with  $\langle P_{||}(t_1)P_{||}(t_2) \rangle$  would have an extra term at  $\omega = 0$  proportional to  $\delta(\omega)$ . Therefore this assumption does not affect the frequency dependence of the noise spectrum for  $\omega \neq 0$ . It does, however, introduce temperature dependence into the noise spectrum.

Putting this all together, we arrive at the following expression for the spectral density of polarization fluctuations of a single TLS:

$$\begin{aligned} \frac{S_{P_{||}}(\omega)}{\langle \delta P_0^2 \rangle} = & \left( \frac{\Delta}{E} \right)^2 [1 - \tanh^2(E/2k_B T)] \frac{T_1}{1 + \omega^2 T_1^2} \\ & + \left( \frac{\Delta_0}{E} \right)^2 \left[ (\rho_{--}) \frac{T_2}{1 + (\omega - E/\hbar)^2 T_2^2} \right. \\ & \left. + (\rho_{++}) \frac{T_2}{1 + (\omega + E/\hbar)^2 T_2^2} \right]. \end{aligned} \quad (52)$$

The first term is the relaxation contribution and the last two terms are the resonance contribution.

In Fig. 2, we show the spectral density  $S_{P_{||}}(\omega)/\langle \delta P_0^2 \rangle$  of polarization fluctuations of a single TLS at low frequencies. We consider three values of the TLS energy splitting as shown in the legend. In the low frequency range, the noise spectrum is dominated by the relaxation

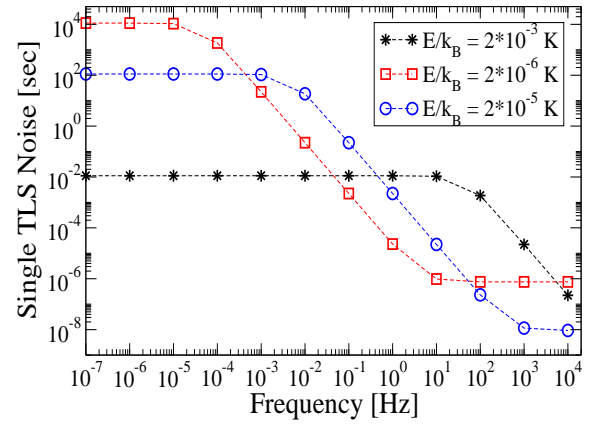


FIG. 2: Log-log plot of the low frequency polarization noise of a single two level system given by Eq. (52) versus frequency for three different values of the TLS energy splitting as shown in the legend. For all 3 cases,  $T = 0.1$  K,  $J/J_c = 10^{-4}$ ,  $\Omega = E/\hbar$ , and  $\Delta = \Delta_0 = E/\sqrt{2}$ .

contribution which is a Lorentzian. Thus the noise spectrum is flat for  $\omega \ll 1/T_1$ . As the frequency increases, it rolls over at  $\omega \sim 1/(T_1)$ , and goes as  $1/\omega^2$  for  $\omega > 1/T_1$ . At very high frequencies ( $\omega > |E/\hbar|$ ), it saturates to a constant (white noise) due to the resonance terms. In addition, we find that the low frequency noise is independent of electromagnetic flux  $J/J_c$  and the angular frequency  $\Omega$  of the driving electric field.

The contribution of the resonance terms to the noise spectrum is negligible at low frequencies ( $\omega \ll E/\hbar$ ) as expected from simple numerical estimates. However these resonance terms become important to the noise spectrum at high frequencies as shown in Fig. 3. The peaks appear when the resonance condition  $\omega = E/\hbar$  is satisfied. The plot in Fig. 3 was obtained for a low value of  $10^{-4}$  for the ratio  $J/J_c$ . No noticeable differences were obtained when increasing the ratio to values as high as  $10^7$  (not shown).

Figure 4 shows the polarization noise power of a single two level system over a broad range of frequencies that cover both the resonance and relaxation contributions. At low frequencies there is a plateau. Between 100 kHz and 1 GHz, the noise spectrum decreases as  $1/\omega^2$  due to the Lorentzian associated with the relaxation contribution. At higher frequencies there is a resonance peak at  $\Omega = E/\hbar$ .

The effect of TLS saturation can be seen at high frequencies in the plot of the noise of a single TLS versus the ratio  $J/J_c$  as shown in Fig. 5. We plotted the spectral density of polarization fluctuations of an individual TLS given by Eq. (52) at a fixed high frequency ( $\omega/2\pi = 9$  GHz) versus  $J/J_c$ . Notice that the noise is constant as long as  $J \ll J_c$ , then decreases when the electromagnetic flux is comparable to the critical flux, and reaches a value which is 2 times smaller than the previous one for



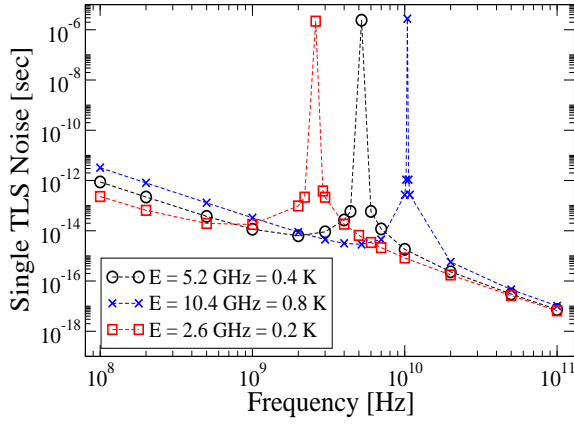


FIG. 3: Log-log plot of the high frequency polarization noise of a single two level system given by Eq. (52) versus frequency for three different TLS energy splittings. The peaks appear when the resonance condition  $\omega = E/\hbar$  is satisfied. In all three cases,  $T = 0.1$  K,  $J/J_c = 10^{-4}$ ,  $\Omega = E/\hbar$  and  $\Delta = \Delta_0 = E/\sqrt{2}$ . No noticeable difference was obtained for  $J/J_c = 10^7 \gg 1$  (not shown).

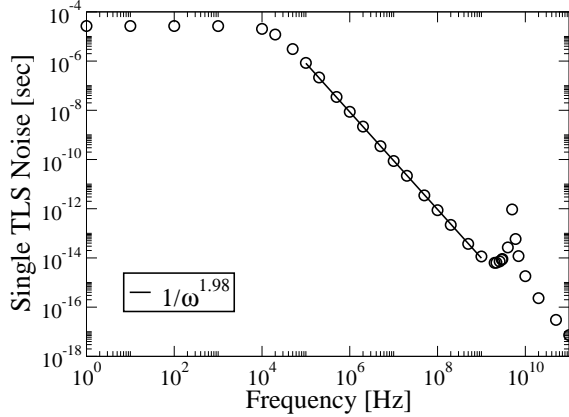


FIG. 4: Log-log plot of the polarization noise of a single two level system given by Eq. (52) versus frequency for a broad range of frequencies.  $E = 5.2$  GHz = 0.4 K,  $J/J_c = 10^{-4}$ ,  $\Omega = E/\hbar$ , and  $\Delta = \Delta_0 = E/\sqrt{2}$ .

$J \gg J_c$ . This is in agreement with our estimates from Section V where we saw that as  $J/J_c$  increases,  $\rho_{--}$  decreases by a factor of 2 from a value close to 1 to a value close to 0.5 when the TLS is saturated.

## B. Polarization and Charge Noise of an Ensemble of TLS Fluctuators

Until now we have analyzed the contribution of a single fluctuator to the polarization noise spectrum. We now average the polarization noise of a single fluctuator over an ensemble of independent fluctuators in a volume  $V_o$ . The distribution function over TLS parameters was given in Section II as  $P(\Delta, \Delta_0) = P_{TLS}/\Delta_0$ . Using this, we

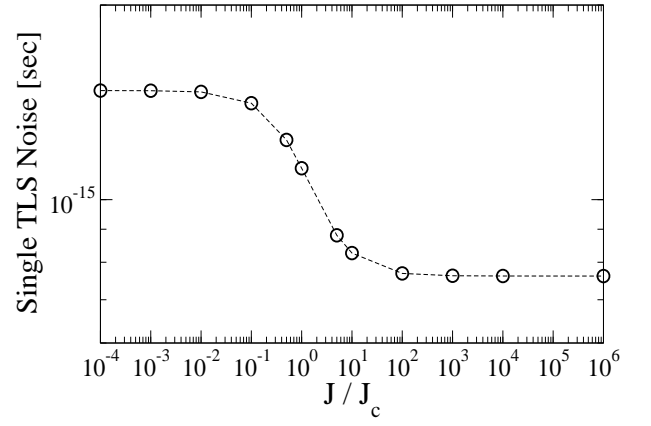


FIG. 5: Log-log plot of the high frequency polarization noise for a single two level system given by Eq. (52) versus  $J/J_c$ . The amplitude of the noise decreases by a factor of two as saturation is achieved for  $J/J_c > 1$ .  $\omega/2\pi = 9$  GHz,  $E/\hbar = \Omega/2\pi = 10$  GHz,  $\Delta = \Delta_0 = E/\sqrt{2}$ , and  $T = 0.1$  K.

obtain:

$$\begin{aligned} \frac{S_{P_{||}}(\omega)}{\langle \delta P_0^2 \rangle} &= V_o \int_{\Delta_{min}}^{\Delta_{max}} d\Delta \int_{\Delta_{0,min}}^{\Delta_{0,max}} d\Delta_0 P(\Delta, \Delta_0) \\ &\times \left( \frac{\Delta}{E} \right)^2 [1 - \tanh^2(E/2k_B T)] \frac{T_1}{1 + \omega^2 T_1^2} \\ &+ \left( \frac{\Delta_0}{E} \right)^2 \left[ (\rho_{--}) \frac{T_2}{1 + (\omega - E/\hbar)^2 T_2^2} \right. \\ &\left. + (\rho_{++}) \frac{T_2}{1 + (\omega + E/\hbar)^2 T_2^2} \right] \\ &\equiv V_o I(\omega; \Omega, T, J/J_c), \end{aligned} \quad (53)$$

where  $I(\omega; \Omega, T, J/J_c)$  is the result of integrating over the distribution of TLS parameters  $\Delta$  and  $\Delta_0$ .

To obtain the charge noise  $S_Q(\omega)$  from the polarization noise, we use Eq. (27) with

$$\langle \delta P_0^2 \rangle = \left\langle \left( \frac{2p \cos \theta_0}{V_o} \right)^2 \right\rangle \quad (54)$$

$$= \frac{4p^2}{3V_o^2}. \quad (55)$$

We obtain

$$\frac{S_Q(\omega)}{e^2} = \frac{4}{9} \left( \frac{p}{eL} \right)^2 V_o P_{TLS} I(f; \Omega, T, J/J_c). \quad (56)$$

We can evaluate Eq. (56) numerically using  $p = 3.7$  D,  $P_{TLS} = 10^{45} (\text{Jm}^3)^{-1}$ ,  $L = 400$  nm,  $A = 40 \times 800$  nm<sup>2</sup>,  $V_o = AL$ ,  $\Delta_{min} = 0$ ,  $\Delta_{max}/k_B = \Delta_{0,max}/k_B = 4$  K, and  $\Delta_{0,min}/k_B = 2 \times 10^{-6}$  K. The results follow.

### 1. Low Frequency Charge Noise

At low frequencies ( $\omega \ll |E/\hbar|$ ), the normalized charge noise spectrum  $S_Q/e^2$  is shown in Fig. 6. It is flat at very



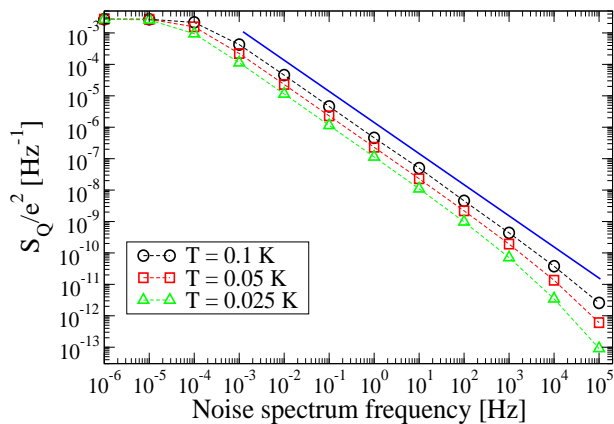


FIG. 6: Log-log plot of the low frequency charge noise  $S_Q/e^2$  averaged over the TLS ensemble versus frequency for different temperatures. The solid line has a slope of  $-1$  corresponding to a perfect  $1/f$  spectrum. Between  $10^{-3}$  and  $10^4$  Hz, the noise spectrum is very close to  $1/f$ . For example, in this frequency range for  $T = 0.1$  K the noise spectrum goes as  $1/(f^{0.998})$ .  $J/J_c = 10^{-4}$ .

low frequencies  $\omega T_1 \ll 1$ . As the frequency increases, it rolls over at  $\omega \approx (T_{1,max})^{-1}$  and decreases as  $1/f$  noise between approximately  $10^{-3}$  and  $10^4$  Hz.

This  $1/f$  noise result agrees with the following simple calculation. Consider an electric dipole moment  $\mathbf{p}$  in a parallel plate capacitor at an angle  $\theta_0$  with respect to the  $z$ -axis which is perpendicular to the electrodes that are a distance  $L$  apart. Assume that the electrodes are at the same voltage. When the dipole flips by  $180^\circ$ , the induced charges on the superconducting electrodes change from  $\mp p \cos \theta_0 / L$  to  $\pm p \cos \theta_0 / L$ . Let  $\delta Q$  denote the magnitude of the charge fluctuations. Then  $\delta Q = |2p \cos \theta_0 / L|$ . Hence the charge of the Josephson junction capacitor produces a simple two state random telegraph signal which switches with a transition rate  $\tau^{-1}$  given by the sum of the rates of transitions up and down. The charge noise spectral density is<sup>26</sup>

$$S_Q^{(i)}(\omega) = 2 \int_{-\infty}^{\infty} d(t_1 - t_2) e^{i\omega(t_1 - t_2)} \langle \delta Q(t_1) \delta Q(t_2) \rangle \\ = (\delta Q)^2 \frac{4w_1 w_2 \tau}{1 + \omega^2 \tau^2}, \quad (57)$$

where the superscript  $i$  refers to the  $i$ th TLS in the substrate or tunnel barrier, and  $w_1$  ( $w_2$ ) is the probability of being in the lower (upper) state of the TLS. In order to average over TLS, we recall that the distribution function of TLS parameters can be written in terms of the energy and relaxation times:<sup>16</sup>

$$P(\tau, E) = \frac{P_{TLS}}{(2\tau \sqrt{1 - \tau_{min}/\tau})}. \quad (58)$$

At low frequencies  $\omega \tau_{min} \ll 1$ , the main contribution to the spectral density comes from slowly relaxing TLS for which  $P(\tau, E) \simeq P_{TLS}/(2\tau)$ . Therefore, the charge noise

of an ensemble of independent fluctuators is

$$S_Q(\omega) \simeq V_o \int_{\tau_{min}}^{\infty} d\tau \int_0^{E_{max}} dE \frac{P_{TLS}}{2\tau} \\ \times \frac{\langle \delta Q^2 \rangle}{\cosh^2(E/2k_B T)} \frac{\tau}{1 + \omega^2 \tau^2}. \quad (59)$$

$\langle \delta Q^2 \rangle$  is the square of the amplitude of charge fluctuations averaged over TLS dipole orientations. We assume that  $\langle \delta Q^2 \rangle$  is independent of  $E$  and  $\tau$ . At low temperatures ( $k_B T \ll E_{max}$ ) and low frequencies ( $\omega \tau_{min} \ll 1$ ), we obtain  $1/f$  charge noise that goes linearly with temperature:

$$\frac{S_Q(\omega)}{e^2} \simeq \frac{2\pi}{3} V_o P_{TLS} k_B T \left( \frac{p}{eL} \right)^2 \frac{1}{\omega}. \quad (60)$$

This agrees with Kogan<sup>26</sup> and Faoro and Ioffe<sup>10</sup>. To estimate the value of  $S_Q/e^2$ , we use  $p = 3.7$  D,  $P_{TLS} \approx 10^{45}$  ( $\text{Jm}^3$ )<sup>-1</sup>,  $L = 400$  nm,  $A = 40 \times 800$  nm<sup>2</sup>, and  $V_o = AL$ . At  $T = 0.1$  K and  $f = 1$  Hz, we obtain  $S_Q/e^2 = 2 \times 10^{-7}$  Hz<sup>-1</sup>, which is comparable to the experimental value of  $4 \times 10^{-6}$  Hz<sup>-1</sup> deduced from current noise.<sup>7</sup> The magnitude of this noise estimate is also in good agreement with our numerical result from Eq. (56), i.e.,  $S_Q(f = 1 \text{ Hz})/e^2 = 2.9 \times 10^{-7}$  Hz<sup>-1</sup>.

Another way to obtain low frequency  $1/f$  noise is the following. At low frequencies the system is in thermal equilibrium, and we can use Eq. (12) from Section III:

$$\frac{S_Q(\omega)}{e^2} = \frac{2k_B T \tan \delta}{e^2 / 2C \omega}. \quad (61)$$

The TLS contribution to the dielectric loss tangent  $\tan \delta$  was calculated by previous workers<sup>11,16,17</sup> who considered fluctuating TLS with electric dipole moments. They found

$$\tan \delta = \frac{\pi p^2 P_{TLS}}{6\epsilon'} \quad (62)$$

where  $\epsilon'$  is the real part of the dielectric permittivity and  $P_{TLS}$  the constant TLS density of states. By plugging Eq. (62) into Eq. (61), we recover Eq. (60).

## 2. High Frequency Charge Noise

Evaluating Eq. (56) numerically at high frequencies yields the normalized charge noise spectrum shown in Fig. 7. We have plotted the contributions coming from the relaxation term and the two resonance terms separately. The relaxation term dominates at low frequencies. It gives  $1/f$  noise at very low frequencies, and transitions to  $1/f^2$  noise when  $\omega > 1/T_{1,min}$ . On the other hand, the resonance terms produce white (flat) noise that dominates at high frequencies. The two curves cross at approximately 0.1 GHz. The inverse of this crossover frequency corresponds to  $T_{1,min} \sim 10^{-9}$  s.

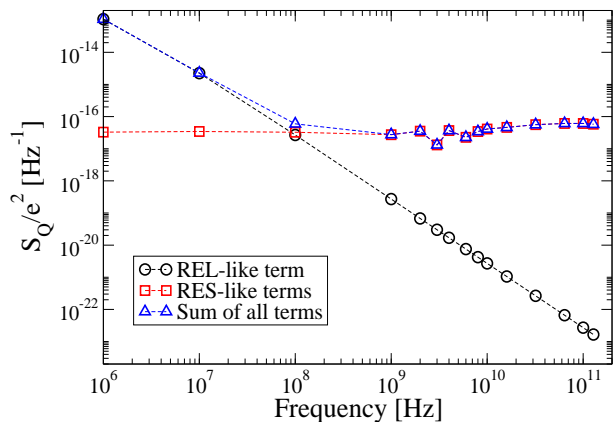


FIG. 7: Log-log plot of the high frequency charge noise  $S_Q/e^2$  averaged over the TLS ensemble versus frequency. The contributions of the relaxation term and the resonance terms are plotted separately. The noise spectrum becomes white noise at frequencies  $\gtrsim 0.1$  GHz. The results were obtained using  $T = 0.1$  K,  $\Omega = 10$  GHz, and  $J/J_c = 10^7$ .

Regarding the temperature dependence, we note that while the low frequency  $1/f$  noise is proportional to temperature, the high frequency white noise amplitude decreases gradually with increasing temperature as shown in Fig. 8. To understand this temperature dependence, note that at high frequencies the resonant terms dominate. These are the last two terms in Eq. (53). The dominant contribution to the integral occurs at resonance ( $\omega = E/\hbar$ ) where the temperature dependence of the integrand goes as  $T_2 \sim 1/T$  and decreases as the temperature increases. However, this decrease is much stronger than seen in the ensemble averaged noise shown in Fig. 8. This may be because in obtaining the ensemble averaged noise, one adds terms away from resonance which have the opposite trend and increase with increasing temperature as  $T_2^{-1} \sim T$ .

In Fig. 9 we show our cumulative numerical results for the charge noise spectrum at both low and high frequencies. As we mentioned previously, there is no noticeable dependence on the electromagnetic flux ratio  $J/J_c$  at either low or high frequencies.

Our result of white noise at high frequencies disagrees with the experiments by Astafiev *et al.*<sup>15</sup> who concluded that the noise increases linearly with frequency. However, we caution that the experiments were done under different conditions from the calculation. The experimentalists<sup>15</sup> applied dc pulses with Fourier components up to a few GHz, possibly saturating TLS with energy splittings in this frequency range. They relied on a Landau-Zener transition to excite the qubit which had a much larger energy splitting, ranging up to 100 GHz, and measured the decay rate  $\Gamma_1$  of the qubit. Then they used  $\Gamma_1$  to deduce the charge noise spectrum at frequencies equal to the qubit splitting by using a formula<sup>27</sup> derived assuming a stationary state. It is not clear whether it is valid to assume a stationary state in the presence of dc

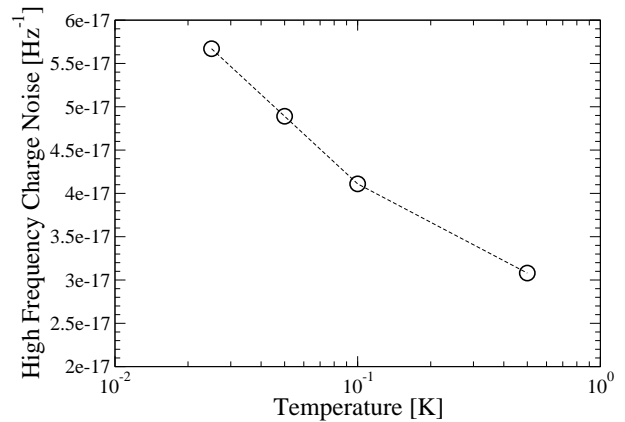


FIG. 8: Plot of the high frequency charge noise  $S_Q(\omega)/e^2$  averaged over the TLS ensemble versus temperature. Notice that the noise decreases gradually with increasing temperature. This plot was obtained for  $\omega/2\pi = 10$  GHz, a driving frequency of  $\Omega/(2\pi) = 10$  GHz, and  $J/J_c = 10^7$ .

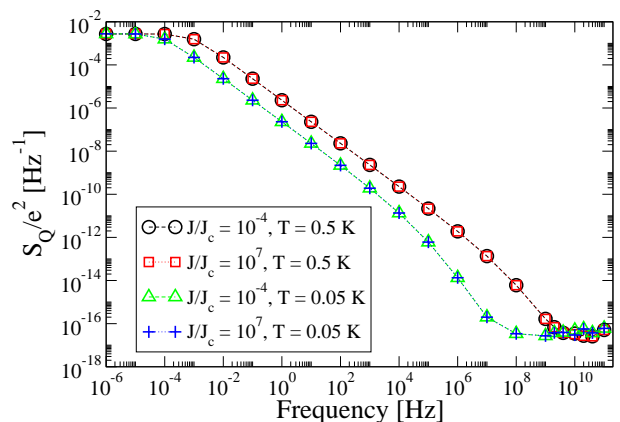


FIG. 9: Log-log plot of the charge noise  $S_Q(\omega)/e^2$  averaged over the TLS ensemble versus frequency. Notice that the results for large and small values of  $J/J_c$  overlap. We have considered two temperatures, 0.5 K and 0.05, respectively, and the driving frequency was  $\Omega/(2\pi) = 10$  GHz.

pulses which lasted for a time ( $\sim 100$  ps) comparable to the lifetime of the qubit.

In contrast, in calculating noise spectra, we make the customary assumption of stationarity. We relate the charge noise spectra to the response to an ac drive for a broad range of frequencies. AC driving of qubits have been used in both theoretical<sup>28-30</sup> and experimental studies<sup>31-34</sup> of qubits. It would be interesting to measure the frequency and temperature dependence of the charge noise of Josephson qubits in the presence of ac driving since no such measurements have been done.

## VII. SUMMARY

To summarize, we have studied the effect of the saturation of TLS by electromagnetic waves on qubit charge noise. Using the standard theory of two level systems with a flat density of states, we find that the charge noise at low frequencies is  $1/f$  noise and is insensitive to the saturation of the two level systems. In addition the low frequency charge noise increases linear with temperature. As one approaches high frequencies, the charge noise plateaus to white noise with a very weak dependence on the driving frequency  $\Omega$  and the ratio  $J/J_c$ .

We found that the high frequency charge noise decreases slightly with increasing temperature.

Finally we note that while we have been considering a Josephson junction qubit, our results on charge and polarization noise have not relied on the superconducting properties of the qubit. So our results are much more general and pertain to the charge noise produced by fluctuating TLS in a capacitor or substrate subject to a driving ac electric field in steady state.

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