Chapter 4
Example of Bragg Law

The spacing of one set of crystal planes in NaCl (table salt) is $d = 0.282$ nm. A monochromatic beam of X-rays produces a Bragg maximum when its glancing angle with these planes is $\theta = 7^o$. Assuming that this is a first order maximum ($n = 1$), find the wavelength of the X-rays. What is the minimum possible accelerating voltage V_0 that produced the X-rays?

The Bragg law is

$$\begin{aligned} 2d\sin\theta &= n\lambda \\ \lambda &= 2d\sin\theta = 2\times(0.282 \text{ nm})\times\sin 7^o \\ &= 0.069 \text{ nm} \end{aligned} \tag{1}$$

To find the minimum voltage V_0, use Duane-Hunt law that says that the kinetic energy of the electrons eV_0 must be at least equal to the energy of the X-ray photon hf:

$$eV_0 \geq hf = \frac{hc}{\lambda} = \frac{1240 \text{ eV} - \text{nm}}{0.069 \text{ nm}} = 18,000 \text{ eV} \tag{2}$$

or

$$V_0 \geq 18,000 \text{ volts} \tag{3}$$

Compton Effect

Usually when light impinges on a system of charges, e.g., an atom or an electron or a metal which has conduction electrons, it is scattered in various directions. Classical electromagnetism explains the scattering by saying that the oscillating electric field of the incident light causes the charges to oscillate, and hence radiate electromagnetic waves in various directions. Recall that oscillating charges produce electromagnetic waves. This is how antennas send out radio waves, for example. Since the charges must oscillate at the frequency of the incident light, the frequency of the scattered and incident waves must be the same:

$$f = f_0 \tag{4}$$

Experiments on light and preliminary experiments on X-rays seemed to confirm this prediction until 1912, when experiments were finding that when high energy (frequency) X-rays scattered from electrons, the scattered frequency f was less than f_0:

$$f < f_0 \tag{5}$$

This was the first evidence of the Compton effect, proposed by the American physicist Arthur Compton in 1923. He also presented experimental evidence for this. The basic idea is that photons carry both energy and momentum. When a photon hits an electron, it's like any other object that collides with the electron: momentum and energy are conserved. So the photon gets scattered by the electron, and the electron recoils, carrying away some of the incident energy and momentum carried by the incident photon. So the outgoing photon has less energy and momentum than the incident photon, and this explains why $f < f_0$.

How much momentum does the photon carry? Start with

$$E^2 = (pc)^2 + \left(mc^2\right)^2 \tag{6}$$

Since $m = 0$ for photons, we have

$$E = pc \tag{7}$$

Now use $E = hf$:

$$p = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda} \tag{8}$$

since $f = c/\lambda$.

As we mentioned above, Compton argued that when the photon collides with an electron, energy and momentum are conserved. He used this to predict the frequency of the scattered photon as a function of the scattering angle θ. He assumed that the X-rays struck stationary electrons, but in fact, the electrons in his target (graphite which is a form of carbon) are contained in atoms. So the electrons have some energy, on the order of eV. And because they are tethered or bound in atoms, they are not free to recoil. But these complications don't matter because the binding energies of the electrons and the energies of the electrons in the outer shells are on the order of eV which is orders of magnitude smaller than the energies of the X-rays. We can estimate the energy of an X-ray. The wavelength of an X-ray is of order 1 $\mathring{A}$ = 0.1 nm which corresponds to an energy of

$$E = hf = \frac{hc}{\lambda} = \frac{1240\text{ eV} - \text{nm}}{0.1\text{ nm}} \approx 10^4\text{ eV} = 10\text{ keV} \tag{9}$$

So X-rays have energies of the order of keV. This is typically the energy of the X-rays used in the dentist's office. Radiation therapy for cancer involves X-rays with MeV energies.

Now let's derive Compton's formula for the wavelength of a scattered X-ray as a function of the scattering angle. Suppose the incident photon has energy $E_0 = hf_0$ and momentum $\mathbf{p}_0$. It hits a stationary electron with rest energy mc^2 and zero momentum. After the collision, the outgoing photon has energy $E = hf$ and momentum $\mathbf{p}$ that makes an angle θ with $\mathbf{p}_0$. The electron recoils with energy E_e and momentum $\mathbf{p}_e$.

Conservation of energy and momentum gives us

$$E_e + E = mc^2 + E_0 \tag{10}$$

and

$$\mathbf{p}_e + \mathbf{p} = \mathbf{p}_0 \tag{11}$$

We treat the electron relativistically so that this will apply to high energies.

$$E_e = \sqrt{(p_e c)^2 + (mc^2)^2} \tag{12}$$

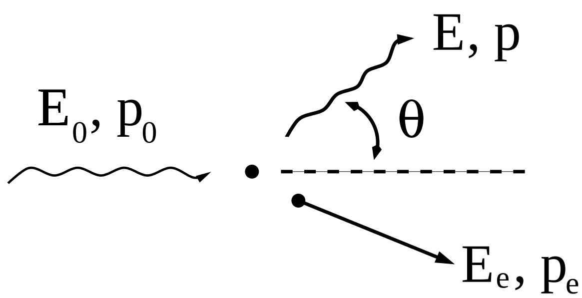

Figure 1: Compton Scattering of an X-ray from a stationary electron.

The incident photon energy is $E_0 = p_0 c$ and the outgoing photon energy is $E = pc$. Since we are interested in the frequency of the outgoing photon, let's eliminate the electron's energy and momentum using these 2 equations. We start with energy conservation:

$$\begin{aligned} E_e &= mc^2 + E_0 - E \\ \sqrt{(p_e c)^2 + (mc^2)^2} &= mc^2 + p_0 c - pc \\ \sqrt{p_e^2 + (mc)^2} &= mc + p_0 - p \end{aligned} \tag{13}$$

Now let's look at momentum conservation:

$$\begin{aligned} \mathbf{p}_e &= \mathbf{p}_0 - \mathbf{p} \\ p_e^2 &= \mathbf{p}_e \cdot \mathbf{p}_e = (\mathbf{p}_0 - \mathbf{p}) \cdot (\mathbf{p}_0 - \mathbf{p}) \\ &= p_0^2 + p^2 - 2\mathbf{p}_0 \cdot \mathbf{p} \\ &= p_0^2 + p^2 - 2p_0 p \cos\theta \end{aligned} \tag{14}$$

We can substitute this into Eq. (13), square both sides, cancel several terms and obtain

$$mc(p_0 - p) = p_0 p(1 - \cos\theta) \tag{15}$$

or

$$\frac{1}{p} - \frac{1}{p_0} = \frac{1}{mc}(1 - \cos\theta) \tag{16}$$

This gives the scattered photon's momentum and from this we can find the scattered photon's frequency and wavelength using

$$E = pc = hf = \frac{hc}{\lambda} \tag{17}$$

So

$$p = \frac{h}{\lambda} \tag{18}$$

or

$$\frac{1}{p} = \frac{\lambda}{h} \tag{19}$$

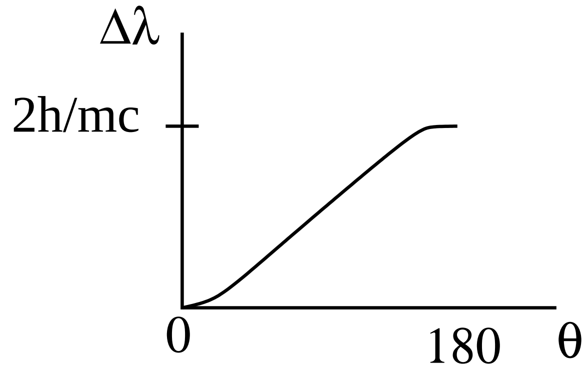

Figure 2: Plot of Compton scattering of an X-ray from a stationary electron as a function of angle θ.

Plugging this into Eq. (16), we obtain

$$\Delta\lambda \equiv \lambda - \lambda_0 = \frac{h}{mc}(1 - \cos\theta) \tag{20}$$

Since $\Delta\lambda \geq 0$, this gives the wavelength increase of the photon scattered through an angle θ. The wavelength is always increased and, hence, the frequency of the scattered photon always decreases. The X-ray loses energy to the recoiling electron. The shift in wavelength is 0 at $\theta = 0$ (forward scattering) and increases as θ increases to a maximum at $\theta = 180^o$. The formula also does not give any shift if there is no electron and the x-ray goes straight through without being scattered ($\theta = 0$). This is what we would expect and is a good sanity check. The magnitude of the shift is

$$\frac{h}{mc} = \frac{hc}{mc^2} = \frac{1240 \text{ eV} - \text{nm}}{0.511 \text{ MeV}} = 0.00243 \text{ nm} \tag{21}$$

In Compton's experiment the incident wavelength was $\lambda_0 = 0.07$ nm. So the predicted shift $\Delta\lambda$ ranged up to 7% of the incident λ_0, and this was a shift that Compton could detect. Compton measured the scattered wavelength at 4 different angles and found convincing confirmation of his theory that photons carry energy and momentum and can be treated like particles.

The formula for $\Delta\lambda$ does not depend on the wavelength itself, so it should be good at any wavelength, including the wavelength of visible light. So why don't we see the shift of visible light? The answer is that the fractional shift $\Delta\lambda/\lambda_0$ is 5000 times smaller for visible light than for X-rays because the wavelength of visible light is 5000 times larger than that of X-rays. (Visible light has wavelengths between 400 nm and 700 nm.) Such a small shift is unobservable. So the longer the incident wavelength, the smaller the Compton effect will be. Also visible light has energies of order eV which is comparable to electron orbital energies and electron binding energies.

Compton also found some scattered X-rays had the same wavelength as the incident wavelength λ_0. These were not X-rays that missed hitting anything. Rather they were X-rays that hit the nucleus or inner electrons (which take several hundred eV to remove), causing the whole atom to recoil. One can go through the same calculation, but with the heavier mass of the entire atom replacing the electron mass. For carbon, the result is a wavelength shift that is 20,000 times smaller than before. This shift is too small to be observed.