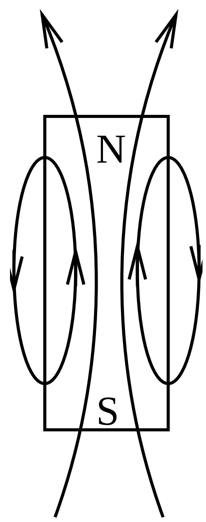
Lectures on Chapter 22: Magnetic Forces and Magnetic Fields

Magnetism

Everyone has played with permanent magnets. You probably know there is a north pole and a south pole on a bar magnet. We can draw magnetic field lines just as we drew electric field lines. The rules: (1) the direction of the tangent to a magnetic field line at any point gives the direction of $\vec{B}$ at that point, and (2) the spacing of the lines is a measure of the magnitude. Lines emerge from N and enter at S.



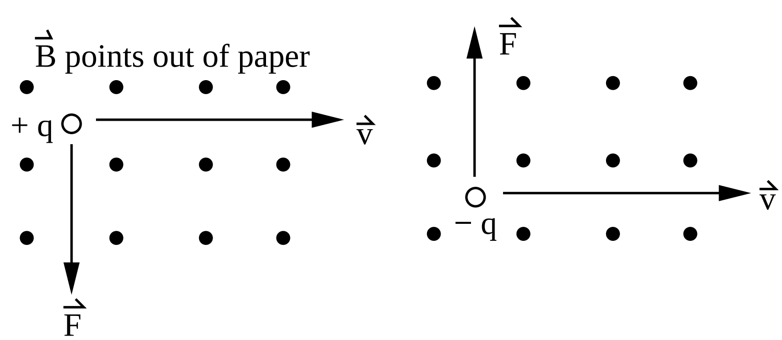
If you have 2 bar magnets, opposite ends attract, i.e., N attracts S. Likes repel, e.g. N repels N. The earth's north pole is a geomagnetic S because field lines enter and point into the earth at the earth's north pole. Antarctica is a magnetic north pole because field lines point up and out of the earth's surface. One way to test to see which way the lines point is to use a compass.

Force Law

What happens if we put a charge $+q$ in a magnetic field $\vec{B}$? If it's sitting still, the answer is nothing. But if it's moving with velocity $\vec{v}$, it feels a force perpendicular to $\vec{v}$ and to $\vec{B}$.

$$\vec{F}_B = q\vec{v} \times \vec{B}$$

q can be positive or negative. $\vec{F}$ changes direction if $q \to -q$. So



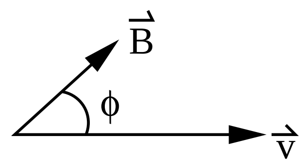
There are several things to notice about $\vec{F}_B$:

1. If $\vec{F}_B \perp \vec{v} \Longrightarrow |\vec{v}|$ doesn't change, only direction of $\vec{v}$ changes in a uniform $\vec{B}$ field. (Acceleration comes from change of direction $\vec{v}$.) So no change in the kinetic energy $\frac{1}{2}mv^2$.

2. If $\vec{v} \parallel \vec{B}$, $\vec{F}_B = q\vec{v} \times \vec{B} = 0$. Particle moving parallel to $\vec{B}$ feels no force.

$$F_B = qvB\sin\phi \qquad\qquad \phi = \text{angle between } \vec{v} \text{ and } \vec{B}$$

$\sin(\phi = 0) = \sin(\phi = 180°) = 0$. This applies if $\vec{v} \parallel \vec{B}$



3. Maximum force of deflection occurs for $\phi = 90° \Longrightarrow \vec{v} \perp \vec{B}$

4. $F_B \propto q$, $F_B \propto v$ — Bigger $q \Longrightarrow$ Bigger F_B; Bigger $v \Longrightarrow$ Bigger F_B

5. Magnetic forces do no work because the force is perpendicular to the direction that the particle moves:

$$\begin{aligned} W_B &= \int \vec{F}_B \cdot d\vec{l} = \int q(\vec{v} \times \vec{B}) \cdot d\vec{l} \\ \text{Use} \quad d\vec{l} &= \vec{v}dt \Longrightarrow W_B = q\underbrace{(\vec{v} \times \vec{B}) \cdot \vec{v}}_{0}\,dt = 0 \end{aligned}$$

Units

$$\begin{aligned} [F_B] &= [qvB] \Longrightarrow [B] = \frac{[F]}{[qv]} \\ [B] &= \frac{N}{C \cdot m/s} = \text{Tesla} = T \qquad (1\,\text{tesla} = 1T = 1\frac{N}{C \cdot (m/s)}) \end{aligned}$$

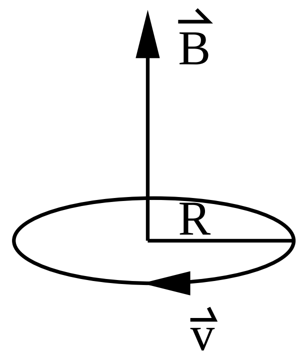
Since $1A = 1C/s \Longrightarrow 1T = 1\frac{N}{C-m/s} = 1\frac{N}{(C/s)-m} = 1\frac{N}{A-m}$
Another common unit of B is the gauss (G):

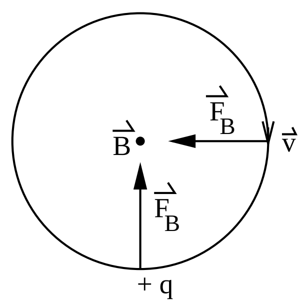
$$1T = 10^4\ G$$

The earth's magnetic field is $\sim 1G \sim 10^{-4}$ T.

Going in Circles

Suppose $\vec{v}$ lies in a plane $\perp$ $\vec{B}$. Here we are considering uniform $\vec{B}$. A charge of $+q$ moves with velocity $\vec{v}$. $\vec{F}_B \perp \vec{v}$ implies that the charge goes in a circle. (Think of whirling a rock around with a string. The force of the string points radially inward.)





Since $\vec{v} \perp \vec{B}$,

$$\vec{F}_B = q\vec{v} \times \vec{B} = qvB(-\hat{r}) \tag{1}$$

Set this equal to the centripetal force:

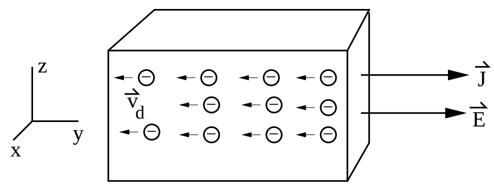
$$qvB = m\frac{v^2}{r} \Longrightarrow r = \frac{mv}{qB} \tag{2}$$

r is the radius of orbit. This equation is called the cyclotron formula. Larger $B \Longrightarrow$ smaller orbit. Larger $v \Longrightarrow$ bigger orbit.

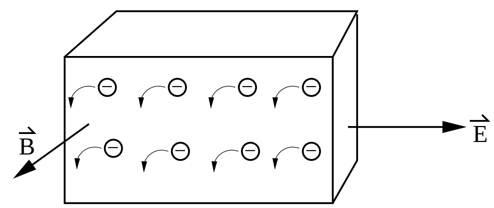
Period $= T =$ circumference/speed $= 2\pi r/v = (2\pi/v)(mv/qB) = 2\pi m/qB$
Notice that the period T has no dependence on the velocity v.
Frequency $= f = 1/T = qB/2\pi m$
Angular Frequency $= \omega = 2\pi f = 2\pi(qB/2\pi m) = qB/m$

Hall Effect

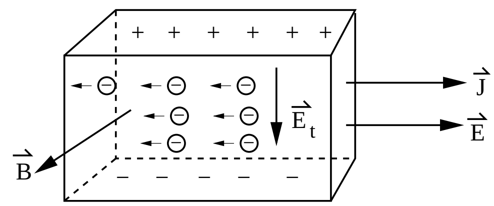
Consider a metal bar with current flowing in it carried by electrons with an average drift velocity $\vec{v_d}$



Now suppose we apply a magnetic field in the $\hat{\imath}$ direction. This initially causes a downward deflection of the moving electrons.



Negative charge builds up at the bottom; positive charge at the top.

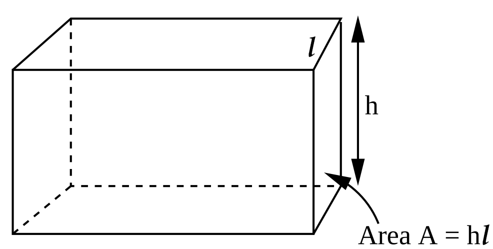


The transverse electric field $\vec{E}_t$ counters the magnetic force so that the electrons again flow in the $-\hat{\jmath}$ direction. Notice that if the charge carriers had been positively charged, $\vec{E}_t$ would point in the opposite direction ($\vec{J}$ in same direction as before). Thus if we measure the voltage difference between top and bottom, the sign tells us the sign of the charge of the carriers.

It is easy to determine the magnitude of $\vec{E}_t$ by balancing the electric force with the magnetic force:

$$qE_t = qv_dB \qquad \Longrightarrow \qquad E_t = v_dB \Longrightarrow v_d = \frac{E_t}{B} \tag{3}$$

We know $J = nqv_d$ implies $n = J/qv_d = JB/qE_t$



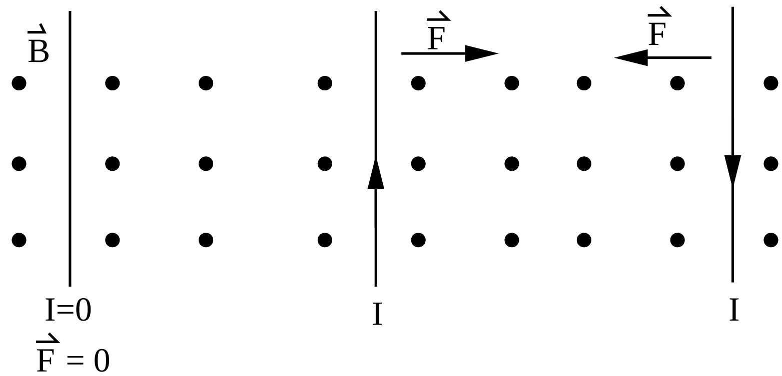
$\Delta V = E_t h \qquad\qquad I = JA$

$$n = \frac{JB}{qE_t}\frac{hA}{hA} = \frac{(JA)B}{q(E_th)}\underbrace{\frac{h}{A}}_{\frac{1}{l}} = \frac{IB}{q(\Delta V)l}$$

Thus we can determine the carrier density n from quantities we can measure.

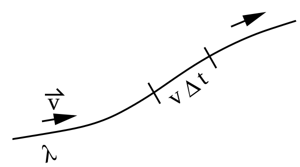
Force on a Current Carrying Wire

Consider a wire carrying current I. If we put the wire in an $\vec{E}$ field, no force acts on it because the wire is neutral. On the other hand, if we put it in a $\vec{B}$ field, there will be a sideways force on the current and hence on the wire.



Remember that I points in the direction of positive charge flow.

Let's be more quantitative about this. Suppose we think of the current as consisting of a charge λ per unit length moving at velocity v down the wire. (Assume that there is a charge density $-\lambda$ that isn't moving so that the wire as a whole is electrically neutral.)



In the time Δt, a length of charge $dl = v\Delta t$ flows past a point. Thus the amount of charge ΔQ that flows past the point is $\Delta Q = \lambda dl = \lambda v \Delta t$. The current I is:

$$I = \frac{\Delta Q}{\Delta t} = \frac{\lambda v \Delta t}{\Delta t} = \lambda v \tag{4}$$

Consider a straight section of wire of length ℓ carrying current I. The amount of moving charge in this section is $q = \lambda\ell$. Thus,

$$\begin{aligned} F &= qvB\sin\phi = (\lambda\ell)vB\sin\phi = \underbrace{\lambda v}_{I}\ell B\sin\phi \\ &= I\ell B\sin\phi \end{aligned}$$

In general, we have

$$\vec{F}_B = I\vec{\ell} \times \vec{B}$$

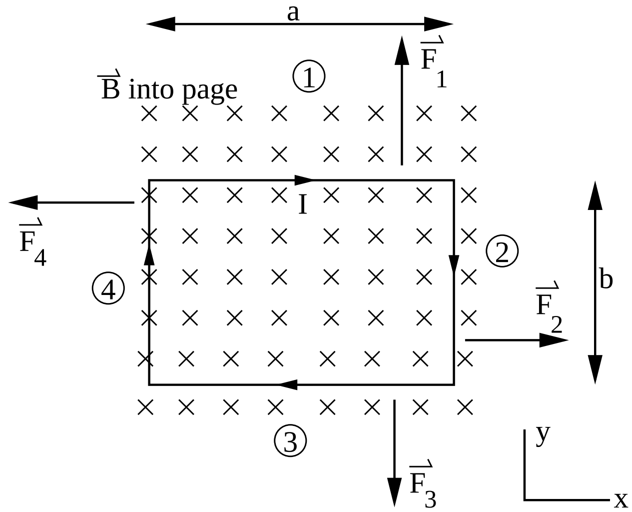
where $\vec{\ell}$ is a length vector which points along the wire. $|\vec{\ell}|$ = length of straight wire segment. If the wire isn't straight, we can divide it up into little straight segments each of length ds. The force on each tiny segment is then

$$d\vec{F}_B = Id\vec{s} \times \vec{B}$$

There is no such thing as an isolated current carrying wire segment. Current coming in one end must go out the other end.

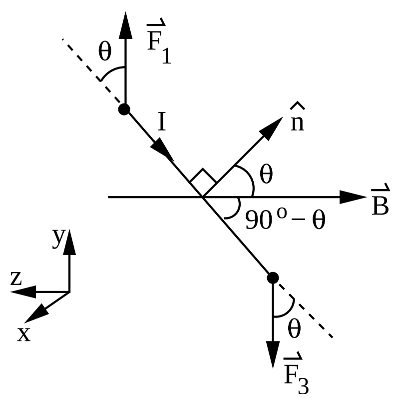
Torque on a Current Loop

Consider a rectangular loop of wire carrying a current in a magnetic field $\vec{B}$. There will be forces on the loop, but the net force will cancel out ($\vec{F}_{\text{tot}} = 0$).



$$\begin{aligned}
\vec{F}_2 &= I\vec{\ell} \times \vec{B} = IbB\hat{x} \\
\vec{F}_4 &= I\vec{\ell} \times \vec{B} = IbB(-\hat{x}) \\
\vec{F}_2 &= -\vec{F}_4 \\
\vec{F}_1 &= I\vec{\ell} \times \vec{B} = IaB\hat{y} \\
\vec{F}_3 &= I\vec{\ell} \times \vec{B} = IaB(-\hat{y}) \\
\vec{F}_1 &= \vec{F}_3
\end{aligned} \tag{5}$$

Now suppose we tilt the loop by rotating it about the x-axis.



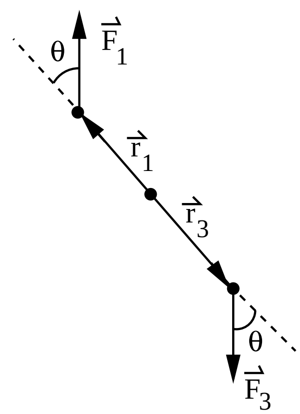
$$\begin{aligned}
\vec{F}_2 &= I\vec{\ell} \times \vec{B} = IbB\sin(90^\circ - \theta)\hat{x} \\
&= IbB\cos\theta\hat{x} \\
\vec{F}_4 &= -\vec{F}_2 \Longrightarrow \text{ Forces on sides 2 and 4 cancel out}
\end{aligned}$$

$\vec{F}_1$ and $\vec{F}_3$ want to rotate the loop so that $\hat{n}$ points along $\vec{B}$. ($\hat{n}$ is a unit vector normal (or perpendicular) to the plane of the loop.) $\vec{F}_1$ and $\vec{F}_3$ produce a torque.

$$\begin{aligned} \vec{F}_1 &= -\vec{F}_3 \qquad\qquad \vec{F}_1 = I\vec{\ell} \times \vec{B} = IaB\hat{y} \\ \vec{\tau} &= \vec{r} \times \vec{F} \\ &= \vec{r}_1 \times \vec{F}_1 + \vec{r}_3 \times \vec{F}_3 \end{aligned}$$

$\vec{r}_1$ and $\vec{r}_3$ are measured from the midpoint.

$$|r_1| = \frac{b}{2} \quad |r_3| = \frac{b}{2}$$
$$\vec{r}_1 = -\vec{r}_3 \quad \text{and} \quad \vec{F}_1 = -\vec{F}_3$$



$$\begin{aligned} \vec{\tau} &= \vec{r}_1 \times \vec{F}_1 + (-\vec{r}_1) \times (-\vec{F}_1) \\ &= 2\vec{r}_1 \times \vec{F}_1 \\ &= 2r_1F_1 \sin\theta \\ &= 2(\frac{b}{2})(IaB)\sin\theta \\ &= IabB\sin\theta \\ &= IAB\sin\theta \qquad\qquad A = ab = \text{area of loop} \end{aligned}$$

If the loop were replaced by a coil with N turns the torque would be

$$\tau_N = N\tau = NIAB\sin\theta \tag{6}$$

Getting a current loop to turn in a magnetic field is the principle behind electric motors. The loop keeps turning because the direction of the current is switched every half cycle. An electric motor is what starts your car.

Magnetic Dipole

We can associate a **magnetic dipole moment** $\vec{\mu}$ with the current loop. The direction of $\vec{\mu}$ is normal to the loop ($\vec{\mu} \parallel \hat{n}$) given by the right hand rule with the fingers in the direction of the current. The magnitude of μ for a flat loop with N turns is

$$\mu = NIA \qquad A = \text{area of loop}$$

Then our previous expression for τ becomes

$$\tau = NIAB\sin\theta = \mu B\sin\theta$$

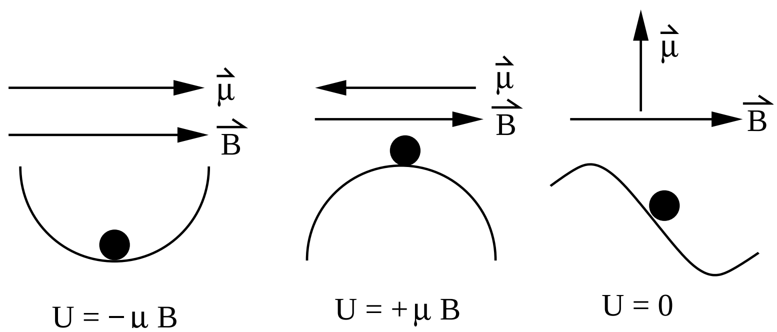
The general vector relation is

$$\vec{\tau} = \vec{\mu} \times \vec{B} \tag{7}$$

The magnetic potential energy is

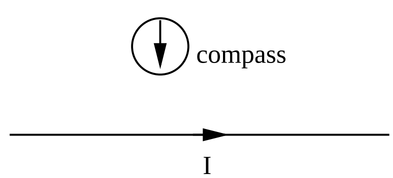
$$U = -\vec{\mu} \cdot \vec{B} \tag{8}$$

The coil aligns itself in the $\vec{B}$ field so as to minimize its potential energy.

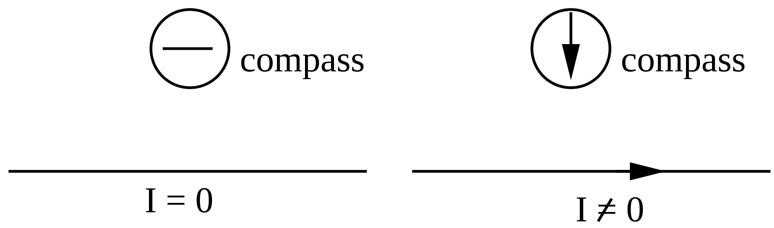


Currents produce magnetic fields

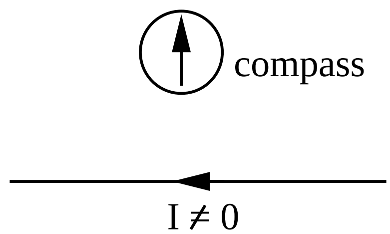
How are magnetic fields produced? One source is permanent magnets (refrigerator magnets). Another turns out to be electric currents. The connection between electricity and magnetism was made in 1819 by Hans Christian Oersted. One day, while teaching class, he brought a compass near a wire with a current flowing through it. Since the needle was perpendicular to the wire, nothing happened.



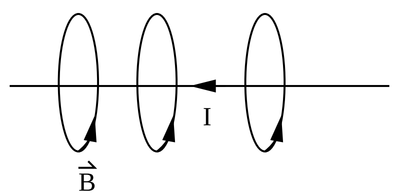
Later, after class, Oersted decided to try it again, but this time with the wire oriented parallel to the compass needle.



The needle turned perpendicular to the wire. When he reversed the current, the compass needle reversed its direction. Oersted had shown that electrical currents produce magnetic fields.



The magnetic field near a wire is much larger than the earth's magnetic field. So we can use a compass to map out the field of a current carrying wire. Denote the magnetic field by $\vec{B}$. $\vec{B}$ circles around the wire. Right hand rule: if thumb points in the direction of current flow, your fingers curl in the direction of the field.



Biot-Savart Law

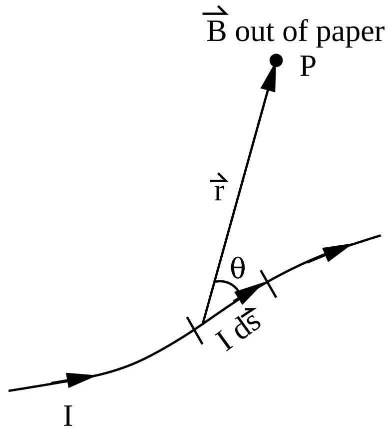
Recall that to find the $\vec{E}$ produced by a continuous charge distribution q, we divide the charge into little pieces dq. Each piece produces $d\vec{E}$ of magnitude

$$|d\vec{E}| = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{dq}{r^2} \tag{9}$$

Similarly, to find the $\vec{B}$ field produced by a current carrying wire, we divide the wire into tiny segments $Id\vec{s}$. $|d\vec{s}|$ = length of segment. $d\vec{s}$ points in the direction of current flow. Note that $I\,d\vec{s}$ is a vector while dq is a scalar.

Suppose a point P is a distance r from $Id\vec{s}$. Then the **magnitude** of the magnetic field set up at point P by the current element Ids is

$$|d\vec{B}| = \frac{\mu_0}{4\pi}\frac{Ids\sin\theta}{r^2}$$



μ_0 is the **permeability constant** (or permeability of free space):

$$\begin{aligned} \mu_0 &= 4\pi \times 10^{-7}\,\frac{T \cdot m}{A} \\ &\approx 1.26 \times 10^{-6}\,\frac{T \cdot m}{A} \end{aligned}$$

μ_0 is the magnetic analog of ε_0. θ is the angle between $d\vec{s}$ and $\vec{r}$. The direction of $d\vec{B}$ is given by $d\vec{s} \times \vec{r}$. $\vec{r}$ points from $d\vec{s}$ to P. Putting this together gives the **Biot-Savart Law:**

$$d\vec{B} = \left(\frac{\mu_0}{4\pi}\right) \frac{I d\vec{s} \times \vec{r}}{r^3}$$

Let $\vec{r} = \vec{r}/r$ be a unit vector in the $\vec{r}$ direction. Then

$$d\vec{B} = (\frac{\mu_0}{4\pi}) \frac{I d\vec{s} \times \hat{r}}{r^2}$$

Recipe for Using the Biot-Savart Law

1. Divide current into tiny pieces $Id\vec{s}$ where $d\vec{s}$ points in the direction of current flow.
2. Find direction of the field $d\vec{B}$ due to $Id\vec{s}$ at a distance $\vec{r}$ away: $d\vec{B} \parallel (d\vec{s} \times \vec{r})$
3. Calculate the magnitude $|d\vec{B}| = \frac{\mu_0}{4\pi} \frac{I ds \sin\theta}{r^2}$
4. Integrate over the entire current flow:

$$\vec{B} = \int d\vec{B} = \frac{\mu_0}{4\pi}\int \frac{Id\vec{s}\times\hat{r}}{r^2}$$

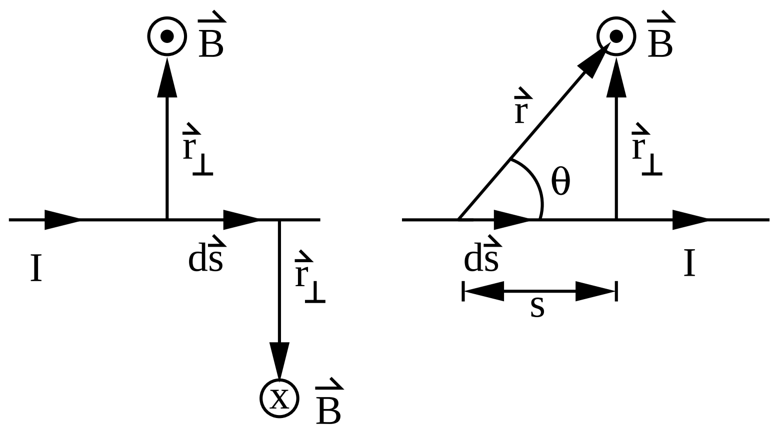
You may have to do this by components:

$$\begin{aligned} B_x &= \frac{\mu_0}{4\pi}\int \frac{(Id\vec{s}\times\hat{r})_x}{r^2} \\ B_y &= \frac{\mu_0}{4\pi}\int \frac{(Id\vec{s}\times\hat{r})_y}{r^2} \\ B_z &= \frac{\mu_0}{4\pi}\int \frac{(Id\vec{s}\times\hat{r})_z}{r^2} \end{aligned}$$

Notice that $d\vec{B}$ falls off as $\frac{1}{r^2}$ just like $d\vec{E}$. Notice also that $d\vec{B} \perp d\vec{s}$ and $d\vec{B} \perp \vec{r}$.

$\vec{B}$ Due to Long Straight Wire

The simplest application of Biot-Savart is to find the $\vec{B}$ field produced by a long straight wire. Notice that our earlier claim of $\vec{B}$ curling around the wire agrees with what we got from $d\vec{s}\times\vec{r}$.



To find the magnitude of $|\vec{B}|$, we use

$$|d\vec{B}| = \frac{\mu_0}{4\pi}\frac{Ids\sin\theta}{r^2} \tag{10}$$

Notice that $d\vec{B}$ has the same direction (out of the paper) for every current element into which the wire can be divided. Thus we can add the magnitudes

$$B = \int dB \qquad \text{integrate over whole wire}$$

Note that $r^2 = s^2 + r_\perp^2$ and $\sin\theta = r_\perp/r = r_\perp/\sqrt{r_\perp^2 + s^2}$

$$dB = \frac{\mu_0}{4\pi}\frac{Ids\sin\theta}{r^2} = \frac{\mu_0}{4\pi}\frac{Ids}{(s^2+r_\perp^2)}\frac{r_\perp}{\sqrt{r_\perp^2+s^2}}$$

$$
\begin{aligned}
&= \frac{\mu_0 I}{4\pi}\frac{r_\perp ds}{(s^2+r_\perp^2)^{\frac{3}{2}}} \\
B &= \int dB = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty}\frac{r_\perp ds}{(s^2+r_\perp^2)^{\frac{3}{2}}} \\
&= \frac{\mu_0 I}{4\pi}\left[\frac{1}{r_\perp}\frac{s}{\sqrt{r_\perp^2+s^2}}\right]_{s=-\infty}^{s=\infty} \approx \frac{\mu_0 I}{4\pi r_\perp}\underbrace{\left[\frac{s}{|s|}\right]_{s=-\infty}^{s=\infty}}_{1-(-1)=2} \\
&= \frac{2\mu_0 I}{4\pi r_\perp} = \frac{\mu_0 I}{2\pi r_\perp}
\end{aligned}
$$

Aside: We had to evaluate the integral $I = \int \frac{r_\perp ds}{(s^2+r_\perp^2)^{\frac{3}{2}}}$. This is how we did this:

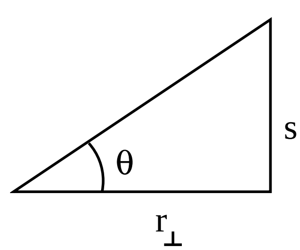
$$
I = \int \frac{r_\perp ds}{(s^2+r_\perp^2)^{\frac{3}{2}}} = \int \frac{r_\perp ds}{r_\perp^2(\frac{s^2}{r_\perp^2}+1)^{\frac{3}{2}}} = \int \frac{ds}{r_\perp^2(1+\frac{s^2}{r_\perp^2})^{\frac{3}{2}}}
$$

Let $\frac{s}{r_\perp} = \tan\phi$ $\quad d(\frac{s}{r_\perp}) = \frac{ds}{r_\perp} = \sec^2\phi d\phi$ $\quad$ because $d\,(\tan\phi) = \sec^2\phi d\phi$

$$
I = \frac{1}{r_\perp}\int\frac{\sec^2\phi d\phi}{(1+\tan^2\phi)^{\frac{3}{2}}}
$$

Use $1+\tan^2\phi = \sec^2\phi$

$$
\begin{aligned}
I &= \frac{1}{r_\perp}\int\frac{1}{r_\perp}\int\frac{\sec^2\phi d\phi}{(\sec^2\phi)^{\frac{3}{2}}} = \frac{1}{r_\perp}\int\frac{\sec^2\phi d\phi}{\sec^3\phi} \\
&= \frac{1}{r_\perp}\int\frac{d\phi}{\sec\phi} = \frac{1}{r_\perp}\int\cos\phi d\phi \\
&= \frac{1}{r_\perp}\sin\phi \\
&= \frac{1}{r_\perp}\frac{s}{\sqrt{s^2+r_\perp^2}}
\end{aligned}
$$



So

$$
\int\frac{r_\perp ds}{(s^2+r_\perp^2)^{\frac{3}{2}}} = \frac{1}{r_\perp}\frac{s}{\sqrt{s^2+r_\perp^2}}
$$

So a long straight wire produces a $\vec{B}$ field

$$B = \frac{\mu_0 I}{2\pi r_\perp}$$

where $r_\perp$ is the perpendicular distance from the wire. Notice that B gets weaker as you go farther from the wire. Also B increases linearly with the current strength I.

$\vec{F}_B$ on a Current Carrying Wire

If we put a long straight current carrying wire in an external magnetic field $\vec{B}_{\text{ext}}$ the total magnetic field $\vec{B}_{\text{tot}}$ is given by

$$\vec{B}_{\text{tot}} = \vec{B}_{\text{ext}} + \vec{B}_{\text{wire}}$$

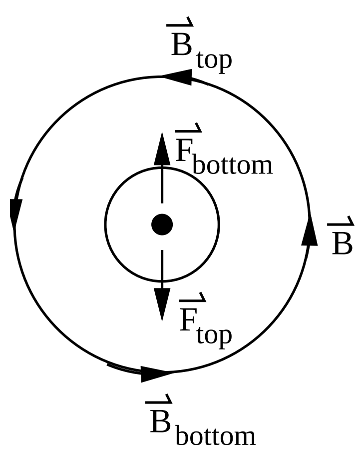
where $\vec{B}_{\text{wire}}$ is due to the current in the wire. However, the force on the wire is due only to $\vec{B}_{\text{ext}}$ because the wire does not exert force on itself. Thus

$$\vec{F}_B = I\vec{\ell} \times \vec{B}_{\text{ext}}$$

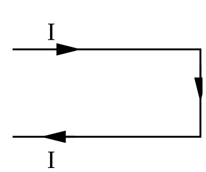
To see why

$$\vec{F} = I\vec{\ell} \times \vec{B}_{\text{wire}} = 0$$

note that $\vec{B}_{\text{wire}}$ on one side of the wire is opposite to that on the other side (e.g. $\vec{B}^{\text{top}}_{\text{wire}} = -\vec{B}^{\text{bottom}}_{\text{wire}}$). So the forces cancel out.



However if the wire was bent, one part of the wire could exert force on the other part.



Force Between 2 Wires

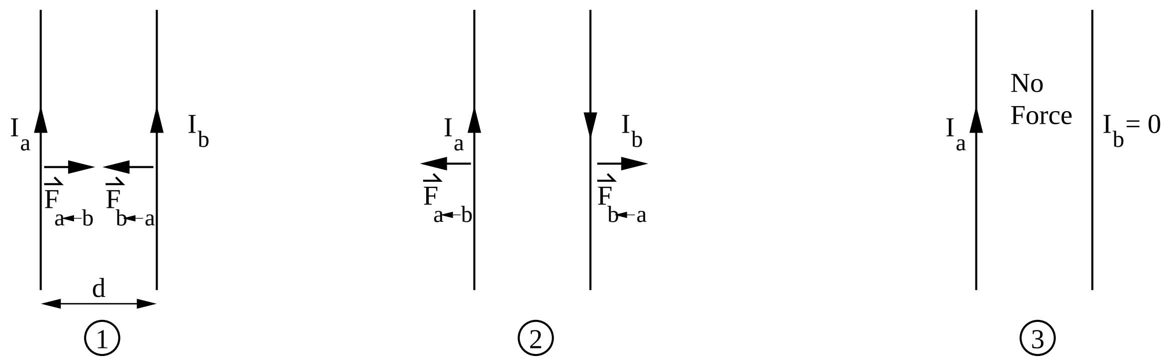
Rather than considering a bent wire (with complications like bends and corners), let's consider 2 parallel wires with currents in them. Then the field $\vec{B}_a$ produced by wire $\underline{a}$ will exert a force $\vec{F}_{b\leftarrow a}$ on wire $\underline{b}$:

$$\vec{F}_{b\leftarrow a} = I_b\vec{\ell}_b \times \vec{B}_a \tag{11}$$

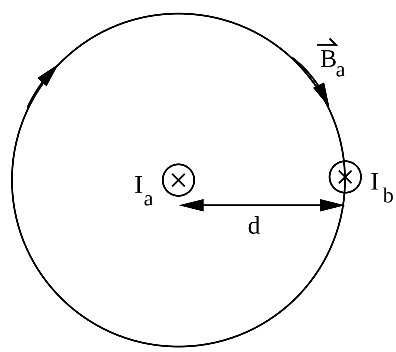
where $\vec{\ell}_b$ = segment of length ℓ_b pointing in direction of I_b. Similarly wire b exerts a force on wire a:

$$\vec{F}_{a\leftarrow b} = I_a\vec{\ell}_a \times \vec{B}_b \tag{12}$$

Let's see which way these forces point:



Let's calculate $\vec{F}_{b\leftarrow a}$ in case 1:



$$\begin{aligned}
\vec{F}_{b\leftarrow a} &= I_b\vec{\ell}_b \times \vec{B}_a & \vec{\ell}_b \perp \vec{B}_a \\
F_{b\leftarrow a} &= I_b\ell_b B_a & B_a = \frac{\mu_0 I_a}{2\pi d} \\
&= I_b\ell_b\frac{\mu_0 I_a}{2\pi d} \\
&= \frac{\mu_0 I_a I_b \ell_b}{2\pi d}
\end{aligned}$$

Notice that the force depends on both I_a and I_b. Similarly,

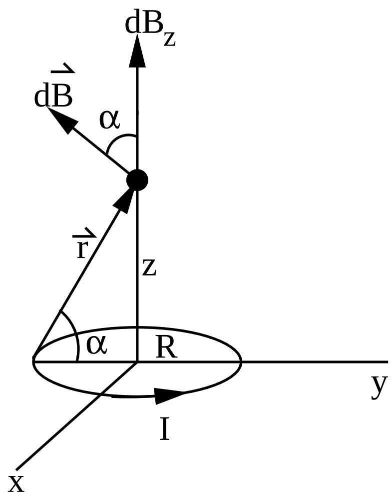
$$F_{a\leftarrow b} = \frac{\mu_0 I_a I_b \ell_a}{2\pi d} \tag{13}$$

The force per unit length is

$$\frac{F_{a\leftarrow b}}{\ell_a} = \frac{\mu_0 I_a I_b}{2\pi d} \tag{14}$$

Current Loop

We can use the Biot-Savart law to find the magnetic field $\vec{B}$ a distance z above the center of a current loop of radius R.



We divide the ring into segments $d\vec{s}$. $d\vec{s} \perp \vec{r}$. So

$$\begin{aligned} dB &= \frac{\mu_0}{4\pi}\frac{I ds \sin 90°}{r^2} \\ &= \frac{\mu_0}{4\pi}\frac{I ds}{r^2} \end{aligned}$$

We want the z-component of dB. (By symmetry, $\vec{B} \parallel \vec{k}$)

$$\begin{aligned} dB_z &= dB \cos\alpha \qquad\qquad \cos\alpha = \frac{R}{r} = \frac{R}{\sqrt{R^2+z^2}} \\ &= \frac{\mu_0}{4\pi}\frac{I\ ds}{r^2}\cos\alpha \\ &= \frac{\mu_0}{4\pi}\frac{I\ ds}{r^2}\frac{R}{r} \\ &= \frac{\mu_0}{4\pi}\frac{IR\ ds}{r^3} \qquad\qquad r = \sqrt{R^2+z^2} \\ &= \frac{\mu_0}{4\pi}\frac{IR\ ds}{(R^2+z^2)^{\frac{3}{2}}} \end{aligned}$$

$$B_z = \oint dB_z = \frac{\mu_0 I}{4\pi}\oint \frac{R ds}{(R^2+z^2)^{\frac{3}{2}}} = \frac{\mu_0 I R}{4\pi(R^2+z^2)^{\frac{3}{2}}}\underbrace{\oint ds}_{2\pi R}$$

$$
\begin{aligned}
&= \frac{\mu_0 I(2\pi R^2)}{4\pi(R^2+z^2)^{\frac{3}{2}}} \\
&= \frac{\mu_0 I R^2}{2(R^2+z^2)^{\frac{3}{2}}} \\
\vec{B} &= \frac{\mu_0 I R^2}{2(R^2+z^2)^{\frac{3}{2}}}\,\hat{k} \qquad\qquad (\vec{B} = B_z\hat{k}) \\
&= \frac{\mu_0 I(\pi R^2)}{2\pi(R^2+z^2)^{\frac{3}{2}}}\,\hat{k} \qquad\qquad \pi R^2 = \text{area of loop}
\end{aligned}
$$

If the loop is replaced by a circular coil with N turns of wire, then

$$\vec{B} = \frac{\mu_0 N I \pi R^2}{2\pi(R^2+z^2)^{\frac{3}{2}}}\hat{k}$$

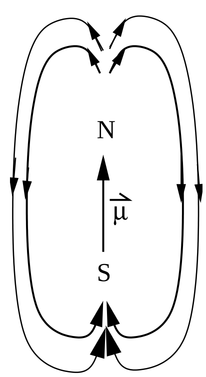
Far above the loop ($z >> R$)

$$\vec{B} \cong \frac{\mu_0 N I \pi R^2}{2\pi z^3}\hat{k}$$

Let's relate this to the magnetic dipole moment $\vec{u}$ of the loop. Recall $\mu = NIA$, where A = area of the loop. The direction of $\vec{\mu}$ is given by the right hand rule: when your fingers curl in the direction of the current, your thumb points in the direction of $\vec{\mu}$. Notice that $\vec{\mu}$ points in the same direction as $\vec{B}$.

$$\vec{B}_(z) = \frac{\mu_0 N I A}{2\pi z^3}\hat{k} = \frac{\mu_0\,\vec{\mu}}{2\pi z^3}$$

where $A = \pi R^2$. This is the magnetic field produced by a dipole moment $\vec{\mu}$ along the z-axis. The field lines are like those of a bar magnet with N and S poles.



Note that our expression for $\vec{B}$ is similar to the electric field $\vec{E}$ produced by an electric dipole $\vec{p}$:

$$\vec{E}_z = \frac{1}{2\pi\varepsilon_0}\frac{\vec{p}}{z^3}$$

Both $\vec{E}$ and $\vec{B}$ fall off as $1/z^3$

Ampere's Law

Recall that in electrostatics, Gauss' law made it easier to find $\vec{E}$ if we were given a charge distribution with symmetry.

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{a} = q_{enc} \tag{15}$$

Ampere's law is the magnetic analog of Gauss' law.

$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0 I_{enc} \tag{16}$$

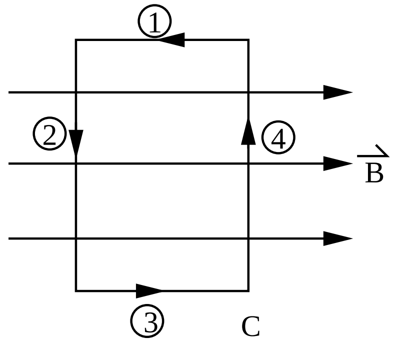
This is Ampere's Law. This tells us that if we encircle a current carrying wire with a closed loop C, the line integral around C will equal $\mu_0 I_{\text{enc}}$. ($d\vec{s}$ does *not* go along the wire.) The direction of C (clockwise vs. counterclockwise) is given by the same right hand rule that gives the direction of $\vec{B}$: thumb along I, fingers curl in direction of C. C can be any shape as long as it doesn't cross itself and it is closed. It is easiest to do the line integral $\oint \vec{B} \cdot d\vec{s}$ if $\vec{B} \parallel d\vec{s}$ or if $\vec{B} \perp d\vec{s}$. Thus Ampere's law allows us to find $\vec{B}$ if the current configurations have symmetry. The following current configurations are commonly evaluated using Ampere's law:

1. Infinitely straight lines
2. Infinite Planes
3. Infinite Solenoids
4. Toroids (Donuts)

Recipe for Using Ampere's Law to Find the $\vec{B}$ Field

1. Determine the direction of $\vec{B}$ produced by the current by symmetry. If the current distribution does not have symmetry, don't use Ampere's law.
2. Draw Amperian loop surrounding the current. Try to make sides of loop such that $d\vec{s} \parallel \vec{B}$ or $d\vec{s} \perp \vec{B}$ on each side.
3. Evaluate line integral $\oint \vec{B} \cdot d\vec{s}$.
4. Evaluate how much current is enclosed by the Amperian loop: I_{enc}.
5. Solve $\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{enc}}$ for the magnitude of $\vec{B}$. (Direction of $\vec{B}$ was determined in step 1.)

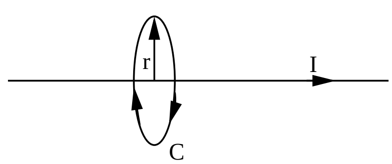
Notice that if $I_{\text{enc}} = 0$, then $\oint \vec{B} \cdot d\vec{s} = 0$. But this doesn't mean that $B = 0$. In the figure $\vec{B} \perp d\vec{s}$ on sides 2 and 4, $\Longrightarrow \int_2 \vec{B} \cdot d\vec{s} = \int \vec{B}_4 \cdot d\vec{s} = 0$.



The line integral for 1 and 3 cancel:

$$\begin{aligned}
\int_1 \vec{B}\cdot d\vec{s} &= B\ell\cos(180^\circ) = -B\ell \\
\int_3 \vec{B}\cdot d\vec{s} &= B\ell\cos(0^\circ) = B\ell \\
\oint \vec{B}\cdot d\vec{\ell} &= \int_1 \vec{B}\cdot d\vec{s} + \int_2 \vec{B}\cdot d\vec{s} + \int_3 \vec{B}\cdot d\vec{s} + \int_4 \vec{B}\cdot d\vec{s} \\
&= -B\ell + 0 + B\ell + 0 \\
&= 0
\end{aligned}$$

Example: Long Straight Wire Find B a distance r from a long straight wire carrying current I. Draw a loop C of radius r around the wire in the direction of $\vec{B}$. $\vec{B}$ circles around the wire. $B = const$ on the loop C because the radius of C is fixed at r.

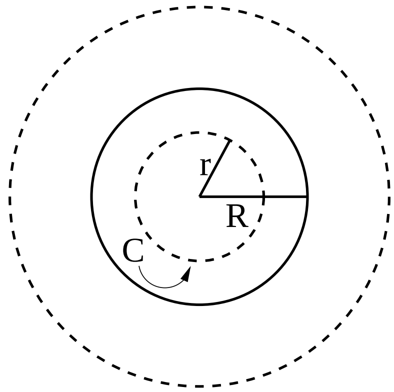


$$\oint_C \vec{B}\cdot d\vec{s} = 2\pi r B$$

$$\oint_C \vec{B}\cdot d\vec{s} = \mu_0 I_{\rm enc} \quad\Longrightarrow\quad \begin{aligned} 2\pi r B &= \mu_0 I_{\rm enc} \\ B &= \frac{\mu_0 I}{2\pi r}\end{aligned}$$

This is the same result that we got using the Biot-Savart law. Notice that this was much easier than our previous calculation.

Suppose the long straight wire has radius R and carries total current I. Find $\vec{B}$ both outside ($r > R$) and inside ($r < R$) the wire. Assume the current is spread uniformly over the cross section of the wire.



For $r > R$, the calculation is the same as above and

$$B = \frac{\mu_0 I}{2\pi r} \qquad r > R$$

For $r < R$ (inside), we draw a loop C of radius r. Then

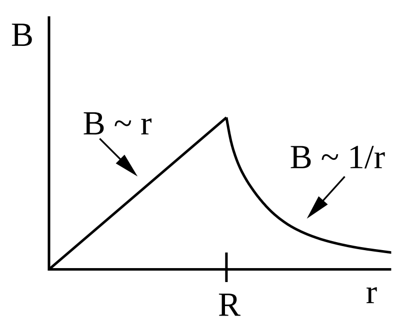
$$\oint_C \vec{B} \cdot d\vec{s} = 2\pi r B$$

$I_{\rm enc}$ = (current density)(area inside loop C) $= J \cdot \pi r^2$. So $J = I/\pi R^2$ which implies

$$I_{\rm enc} = J\pi r^2 = \frac{I}{\pi R^2} \cdot \pi r^2 = \frac{Ir^2}{R^2} \tag{17}$$

$$\begin{aligned} \oint \vec{B} \cdot d\vec{s} &= \mu_0 I_{\rm enc} \\ 2\pi r B &= \mu_0 \frac{Ir^2}{R^2} \\ B &= \frac{\mu_0}{2\pi r}\frac{Ir^2}{R^2} \\ &= \frac{\mu_0 I r}{2\pi R^2} \end{aligned}$$

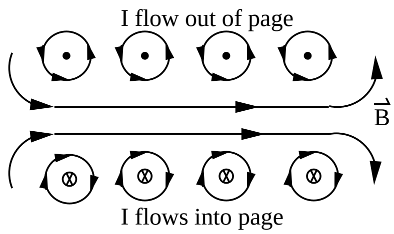
Notice that B increases linearly with r inside the wire.



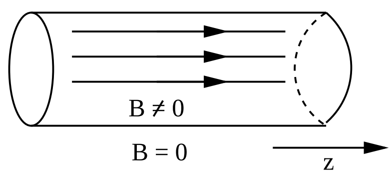
$\vec{B}$ points azimuthally both inside and outside the wire.

Solenoid

A solenoid is a cylindrical coil of wire. We are interested in long straight solenoids whose length L is much longer than the radius ($L >> R$). An infinite solenoid with tightly packed turns of wire has a uniform $\vec{B}$ field inside the coil along the axis and $B = 0$ outside the coil. You can see this by looking at a diagram.

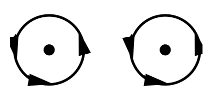


The field between neighboring wires tends to cancel. Outside the solenoid, the field from the top ($\odot\odot$) tends to cancel that from the bottom ($\otimes\otimes$). Inside the top and bottom add

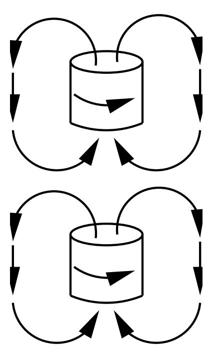


We can systematically deduce the direction of $\vec{B}$. Let's use cylindrical coordinates (r, ϕ, z).

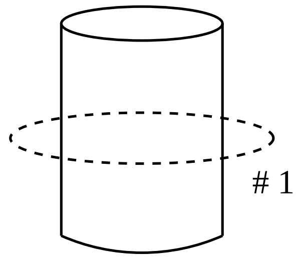
$\mathbf{B_r}$: Is there a radial component B_r? No. Suppose there were an outward pointing radial component ($B_r > 0$). Then if we reverse the current, B_r would point inward. But reversing the current is equivalent to turning the solenoid upside down which shouldn't change B_r. So $B_r = 0$. Another way to see this is to note that B_r from adjacent turns of wire cancel.



Or consider $z = 0$, say, and note that the current from $\pm z$ produce contributions to $B_r(z = 0)$ which tend to cancel.



$\mathbf{B}_\phi$: What about B_ϕ? Take an Amperian loop #1 around the cylinder. By symmetry, $B_\phi = const$ on this loop. The loop has a radius r.

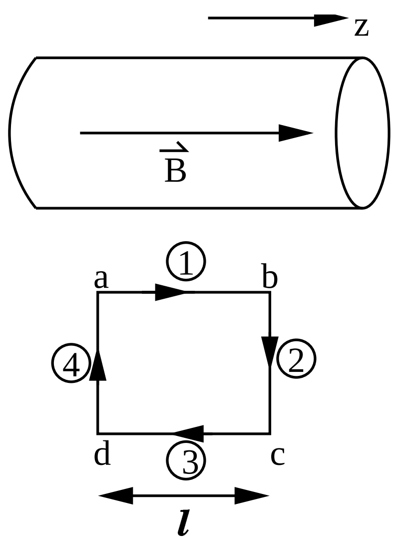


$$\oint_{\#1} \vec{B} \cdot d\vec{s} = B_\phi (2\pi r) = \mu_0 I_{\text{enc}}$$

$I_{\text{enc}} = 0$ because the no current pokes through the surface spanned by loop #1.

$$\Longrightarrow B_\phi (2\pi r) = 0$$
$$\Longrightarrow B_\phi = 0$$

$\mathbf{B_z}$ Only B_z is left. From the right hand rule, $\vec{B}$ points along $+z$ inside. (Same use of right hand rule that you used to get the dipole moment $\vec{\mu}$.) What is B_z outside? We can use Ampere's law to prove that $B = 0$ outside. Draw an Amperian loop outside the solenoid. $I_{\text{enc}} = 0$.



$$\Longrightarrow \oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{enc}} = 0$$

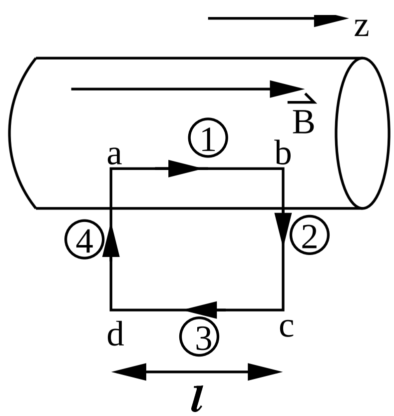
$$\oint \vec{B} \cdot d\vec{s} = \int_a^b \vec{B}_1 \cdot d\vec{s} + \int_b^c \vec{B}_2 \cdot d\vec{s} + \int_c^d \vec{B}_3 \cdot d\vec{s} + \int_d^a \vec{B}_4 \cdot d\vec{s}$$

$\vec{B}_2 \perp d\vec{s}$ and $\vec{B}_4 \perp d\vec{s}$ because $\vec{B} \parallel \hat{z}$ and $d\vec{s} \perp \hat{z}$. $\vec{B}_1$ is constant along side 1; $\vec{B}_3$ is constant along side 3. Let's assume $\vec{B}_1 = B_1\hat{z}$ and $\vec{B}_3 = B_3\hat{z}$ where $B_1 > 0$, $B_3 > 0$; i.e., $\vec{B}_1$ and $\vec{B}_3$ point in the $+\hat{z}$ direction. $\vec{B}_3$ is antiparallel to $d\vec{s}$. So $\vec{B}_3 \cdot d\vec{s} = -B_3 ds = B_3 dz$ since $d\vec{s} = -d\vec{z}$.

$$\begin{aligned}
\oint \vec{B} \cdot d\vec{s} &= \int_a^b B_1 ds - \int_c^d B_3 ds \\
&= \int_a^b B_1 dz + \int_c^d B_3 dz \\
&= B_1 \int_a^b dz + B_3 \underbrace{\int_c^d dz}_{=d-c=-\ell} \\
&= B_1\ell - B_3\ell \\
&= 0 \Longrightarrow B_1 = B_3 = \text{const}
\end{aligned}$$

But infinitely far away from the solenoid, $B = 0$ which implies that the constant $= 0$. So $B_1 = B_3 = 0 \Longrightarrow \quad \vec{B} = 0$ outside the solenoid.

What is B_z inside the solenoid? Draw an Amperian loop as shown. If there are N turns per unit length, then $I_{\text{enc}} = N\ell I$.



Look at $\oint \vec{B} \cdot d\vec{s}$. $\vec{B} \perp d\vec{s}$ on sides 2 and 4. $\vec{B} = 0$ along side 3. So

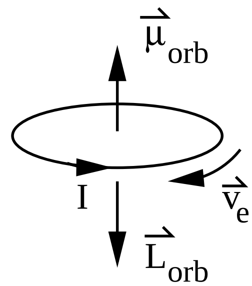
$$\begin{aligned}
\oint \vec{B} \cdot d\vec{s} &= \int_1 \vec{B} \cdot d\vec{s} + \int_2 \vec{B} \cdot d\vec{s} + \int_3 \vec{B} \cdot d\vec{s} + \int_4 \vec{B} \cdot d\vec{s} \\
&= B \int_a^b dz \qquad (\vec{B} \text{ is constant along 1.}) \\
&= B\ell \\
\oint \vec{B} \cdot d\vec{s} &= \mu_0 I_{\text{enc}} \\
B\ell &= \mu_0 N I \ell \\
B &= \mu_0 N I \qquad \text{inside the solenoid} \\
\vec{B} &= \mu_0 N I \hat{z}
\end{aligned}$$

Magnetism in Matter

Electrons can generate magnetic fields in three ways.

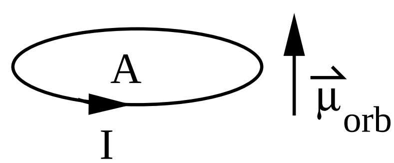
1. Currents in conductors.
2. Orbital motion in an atom.
3. Intrinsic "spin" which produces an intrinsic magnetic moment $\vec{\mu}_B$.

Electrons orbiting around the nucleus of an atom are like a tiny current loop. Associated with this current loop is a magnetic dipole moment $\vec{\mu}_{orb}$. The circular motion of the electrons is also associated with an orbital angular momentum $\vec{L}_{orb}$. It turns out that $\vec{\mu}_{orb} \propto \vec{L}_{orb}$.



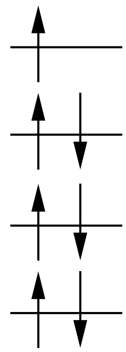
Recall that the magnetic moment produced by a current loop is given by

$$\mu_{orb} = I\,A \tag{18}$$



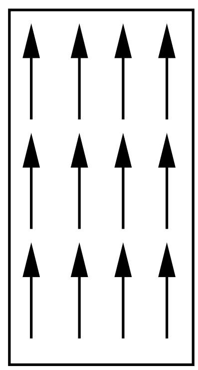
The current $I \sim 1.6 \times 10^{-3}$ A and the area $A = \pi R^2$ where $R \sim 1$ Å. In most substances the orbital magnetic moment of one electron in an atom is cancelled by that of another electron orbiting in the opposite direction. So orbital motion produces very little, if any, magnetism in materials.

The other way electrons produce magnetism is via their intrinsic angular momentum or **spin**. In atoms or ions with many electrons, the electrons fill the orbitals in pairs with opposite alignments of their spins. As a result each pair has zero magnetic moment. But an atom with an odd number of electrons has at least one unpaired electron and a corresponding spin magnetic moment. A substance consisting of these magnetic atoms can exhibit different types of magnetic behavior.



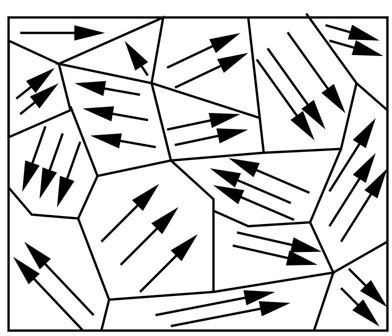
Electrons in atomic energy levels

Perhaps the most well known such behavior is **ferromagnetic**. Iron, cobalt, nickel, gadolinium, and dysprosium are ferromagnets. Ferromagnets are what you use to tack messages to your refrigerator door. In a ferromagnet the magnetic moments of the atoms tend to align parallel to one another.



Ferromagnet

In ferromagnets there are microscopic regions called *domains* within which the moments are aligned parallel to one another. But a given domain can have its net magnetic moment pointing in a different direction from some other domain. All these misaligned domains yield a net magnetic moment of 0. This is why a piece of iron (like a nail) will usually not be magnetic.



But if you put the iron in a strong magnetic field, the domains aligned along the field will grow at the expense of the domains which are not aligned along the field. As a result the iron will acquire a net magnetization and will become magnetized. Magnetization is the magnetic moment per unit volume.