Lectures on Chapter 20: Electric Potential and Capacitance

Gravitational force and electrostatic force are similar:

$$F_{\text{grav}} = G\frac{m_1 m_2}{r^2} \qquad\qquad F_e = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$$

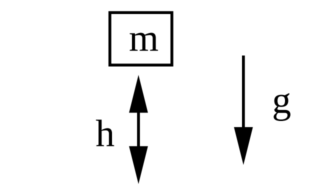
Recall that we can define a gravitational potential energy. If the work done to lift an object from $\underline{a}$ to $\underline{b}$ is

$$W_{ab} = \int_a^b \vec{F}_{\text{grav}} \cdot d\vec{s}$$

then the potential energy gained by the object is

$$\Delta U = U_b - U_a = -W_{ab} = -\int_a^b \vec{F}_{\text{grav}} \cdot d\vec{s}$$

For example, a mass m a distance h above the ground has potential energy



$$\Delta U = +mgh \tag{1}$$

The energy is measured with respect to the ground ($U = 0$ at $h = 0$), but we could put the zero of energy anywhere. It would just add a constant to U_a and U_b but wouldn't change the potential energy difference $U(h) - U(\text{ground})$.

Since gravity is a conservative force, $U_b - U_a$ is independent of the path taken between a and b. All that matters is the endpoints.

In analogy with this, we can define the change in the electric potential energy.

$$\Delta U = -W_{ab} = -\int_a^b \vec{F}_e \cdot d\vec{s}$$

If we have a charge q that finds itself in an electric field $\vec{E}$, then it feels a force

$$\vec{F}_e = q\vec{E}$$

$\vec{E}$ is due to other charges, e.g., another point charge. The electric potential energy associated with dragging the charge from a to its present position b is the negative of the work done in getting it there.

$$\Delta U = -W_{ab} = -\int_a^b \vec{F}_e \cdot d\vec{s} = -\int_a^b q\vec{E} \cdot d\vec{s}$$

$d\vec{s}$ goes along the path we took from a to b. Like gravity, $\vec{F}_e$ is a conservative force so that ΔU depends only on the endpoints a and b, not on the path between them.

Notice that the sign of ΔU will change in $q \to -q$. So if the field comes from positive charges and $q > 0$, we do work in dragging q toward the charges and ΔU increases. If $q < 0$, then q is attracted to the positive charges and drags us toward them; ΔU decreases and is negative. (It's like the difference between uphill and downhill.)

Let's take $q > 0$. Notice that the bigger q is, the more work we do in dragging it around, and the bigger the potential energy is. Remember how it was convenient to divide $\vec{F}_e$ by q to get $\vec{E}$.

$$\vec{E} = \frac{\vec{F}_e}{q} \tag{2}$$

$\vec{E}$ is independent of q which feels it but doesn't produce it, e.g., consider the force produced by a point charge q_1.

$$F_{2\leftarrow 1} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q}{r^2} \tag{3}$$

depends on q and q_1, but

$$\vec{E} = \frac{\vec{F}}{q} = \frac{1}{4\pi\varepsilon_0} \frac{q_1}{r^2} \tag{4}$$

depends only on q_1.

Similarly, it is convenient to divide U by q to get something independent of q.

$$\Delta V = \frac{\Delta U}{q} \quad \text{potential energy per charge}$$

V is called the **electric potential**.

$$\Delta V = \frac{\Delta U}{q} = -\int_a^b \frac{1}{q}\vec{F}_e \cdot d\vec{s} = -\int_a^b \vec{E} \cdot d\vec{s}$$

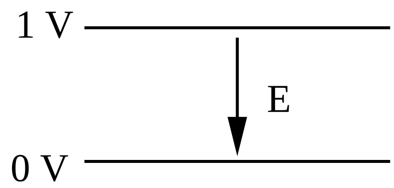
This depends on only on the charges that produce $\vec{E}$. Just as $\vec{E}$ was the force felt by a unit charge ($q = 1$), ΔV is the potential energy of a unit charge, i.e., ΔV is the potential energy per unit charge.

Units

$[\Delta U]$ = joules (after all, it's an energy)
$[\Delta V] = \frac{\text{joules}}{\text{coulomb}}$ = volts

So electric potential ΔV and electric potential energy are different. Sometimes the electric potential is referred to as **voltage**.

Notice that if there is potential difference $\Delta V = 1$volt, the potential energy gained by moving a proton with charge $+e$ through this difference ("uphill" or against the field) is $\Delta U = e\Delta V = 1eV = 1$ electron volt.



An eV is a unit of energy:

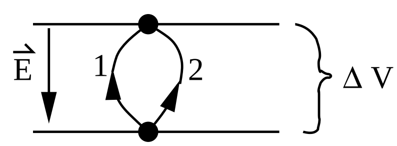
$$1\text{eV} = e\ (1V) = (1.60 \times 10^{-19}C)(1\frac{J}{C}) = 1.60 \times 10^{-19}J$$

In general, moving a charge q through a voltage drop ΔV changes its potential energy by ΔU.

$$\Delta U = q\Delta V$$

So a big voltage can give a small ΔU if q is small. For example, rubbing your hair with a balloon can charge the balloon to several thousand volts but the amount of charge is small ($\sim 10^{-6}$ C) so ΔU is small $\Longrightarrow$ you won't get electrocuted.

The gravitational analog is $U_{\text{grav}}/m = gh$. Just as with gravity, $\Delta V = V_b - V_a$ depends only on where a and b are, not on the path between them. Thus it doesn't matter whether you take path 1 or path 2, ΔV is the same. (The gravitational analog of $E = \frac{F}{q}$ was $F/m = g =$ acceleration.)



You can think of $\frac{U_{\text{grav}}}{m} = gh$ as gravitational potential (if you multiply by m, you get gravitational potential energy). Think of different distances up a hill. Higher points on the hill have greater gravitational potential. Going uphill is like acquiring a higher voltage by dragging a positive charge opposite to the field. "Danger: High Voltage" are like "Danger: Falling rocks" signs.

Equipotential Surfaces

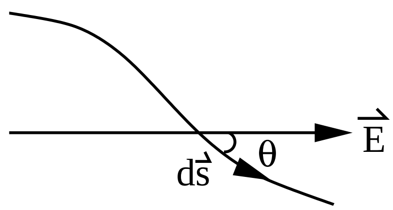
An equipotential surface is the set of points which all have the same electric potential or voltage. These points are usually a surface.

$$\Delta V = V_b = V_a \quad (5)$$

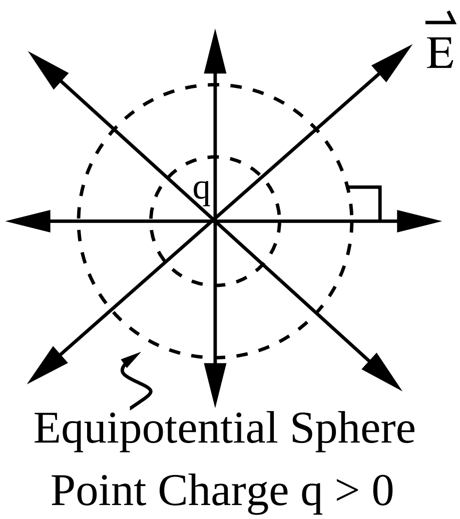
If a and b are on the same equipotential surface, then $V_b = V_a$ and $\Delta V = 0$. Look at

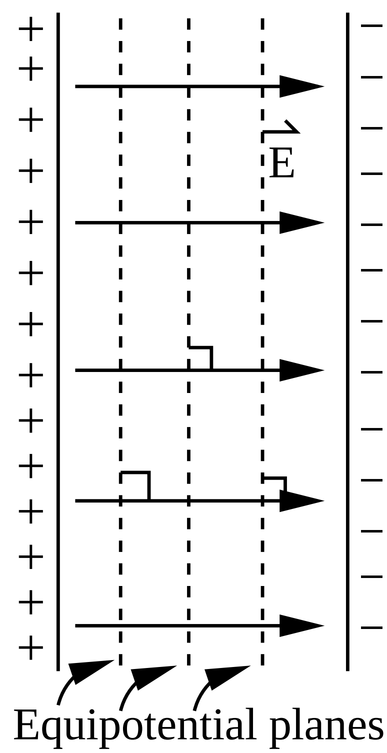
$$\Delta V = V_b - V_a = -\int_a^b \vec{E} \cdot d\vec{s} \tag{6}$$

How can $\Delta V = 0$? One way: $\vec{E} = 0$ everywhere along $d\vec{s}$ (like inside a conductor). Another way: $\int \vec{E} \cdot d\vec{s} = 0$. $\int \vec{E} \cdot d\vec{s}$ is a line integral along the path parameterized by $\vec{s}$. $\int \vec{E} \cdot d\vec{s}$ adds up all the components of $\vec{E}$ along $d\vec{s}$. $\vec{E} \cdot d\vec{s} = |\vec{E}||d\vec{s}| \cos\theta$. If $\vec{E} \perp d\vec{s}$, then $\theta = \frac{\pi}{2} = 90°$, $\vec{E} \cdot d\vec{s} = 0$. So $\Delta V = 0$ on surfaces $\perp \vec{E}$.

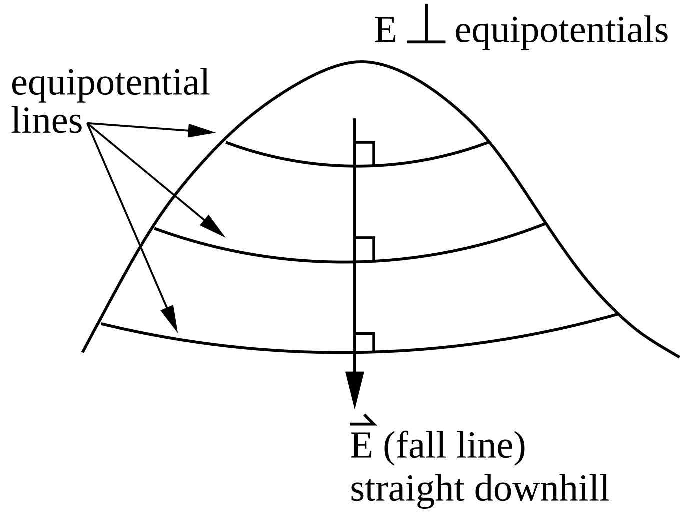


Examples:





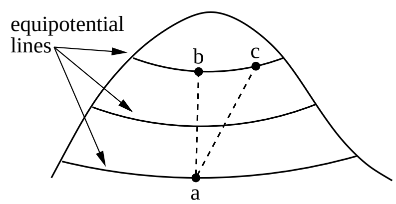
All points on the surface have the same potential but different surfaces have different potentials. Gravitational analogy: Every point at the same altitude h has the same gravitational potential gh. $\vec{F}_{\text{grav}} \perp$ to equipotential surfaces. $\vec{F}_{\text{grav}}$ points along the fall line on a slope.



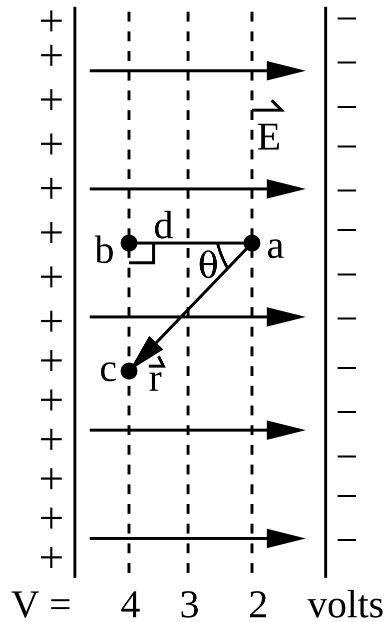
Bigger $E \Longrightarrow$ Bigger V (Swim upstream)
$\Downarrow$ Lots of charge
Charges set up $\vec{E}$. They also set up V. Note ΔV is different from U.

Potential in a Uniform Field

Electric potential is the analog of height on a hill. Notice that the height difference between points a and b is the same as the height difference between point b and c.



Similarly the potential difference between points a and b is the same as the potential difference between points b and c in the figure of equipotential planes in a uniform electric field. Dragging a mass uphill corresponds to dragging a positive charge from a to c or from a to b.



The potential difference between points a and c is given by

$$\Delta V = V_c - V_a = -\int_a^c \vec{E}\cdot d\vec{s} = -\vec{E}\cdot\int_a^c d\vec{s} = -\vec{E}\cdot\vec{r} = -Er\cos\theta = -Ed = V_b - V_a \quad (7)$$

We can take $\vec{E}$ out of the integral because the field is constant. It's a uniform field that is the same everywhere. So we see that in a uniform field the potential is E times the perpendicular distance between equipotential surfaces.

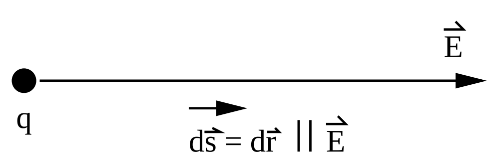
Dimensional analysis: Notice that $[V] = [Ed]$ implies that voltage or potential has the same units as electric field times length or (Newtons/Coulomb) $\times$ meters = Joules/Coulomb=Volts. Similarly, $[E] = [V/d]$ which means that electric field has units of Volts/meter = V/m. Earlier we learned that $[E] = [Force/Charge]$ = Newtons/Coulomb. So 1 V/m = 1 N/C.

Potential of a Point Charge

Consider a point charge $q > 0$ at the origin. Infinitely far away, a test charge q_0 feels no force $\Longrightarrow \vec{E} = 0$ at ∞. So let's set $U_\infty = 0$ and $V_\infty = 0$ because there is no potential energy and no potential at $r = \infty$. What is the potential $V(r)$ a distance r away from q? If we start at r and push $q_0 > 0$ to infinity, then

$$\Delta V = V(\text{end}) - V(\text{start}) = V(\infty) - V(r) = -V(r) < 0$$

This is less than 0 because we lose potential energy; we are going downhill.



$$
\begin{aligned}
\Delta V &= -\int_r^\infty \vec{E}\cdot d\vec{s} \\
&= -\int_r^\infty \vec{E}\cdot d\vec{r} \\
\vec{E} &= \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \Longrightarrow \vec{E} \parallel d\vec{r} \Longrightarrow \vec{E}\cdot d\vec{r} = Edr \\
\Delta V &= -V(r) = -\int_r^\infty Edr' = -\frac{1}{4\pi\varepsilon_0}\int_r^\infty \frac{q}{r'^2}dr' \\
&= -\frac{1}{4\pi\varepsilon_0}\left[-\frac{q}{r'}\right]_{r'=r}^{r'=\infty} \\
&= \frac{1}{4\pi\varepsilon_0}\left[0-\frac{q}{r}\right] \\
-V(r) &= -\frac{1}{4\pi\varepsilon_0}\frac{q}{r} \\
V(r) &= \frac{1}{4\pi\varepsilon_0}\frac{q}{r} > 0 \qquad (8)
\end{aligned}
$$

If $q \to -q$, $V(r) = -\frac{1}{4\pi\varepsilon_0}\frac{q}{r} < 0$. So V changes sign if q sign. $V(r)$ falls off slower than $E \sim 1/r^2$. Notice that $V(r) > 0$ which means that if we started at $r = \infty$ and dragged a charge q_0 to the point r, its potential energy would increase:

$$
U(r) = q_0 V(r) = \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r}
$$

One of the great benefits of V is that it is a scalar, so we don't have to find the components of a vector. $V(r)$ is a scalar field, i.e., a number is associated with every point in space.

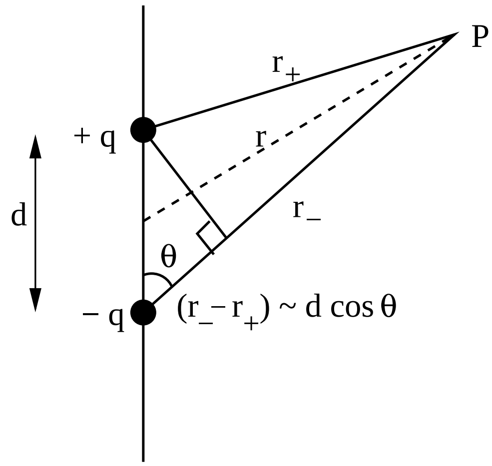
Superposition

To find the potential of a group of charges at a point $\vec{r}$, add the potential r at $\vec{r}$ due to each charge:

$$
V = \sum_{i=1}^{n} V_i = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i}
$$

Because V is a scalar, we can add magnitudes. We don't have to worry about vector components.

Potential of an Electric Dipole



$$
\begin{aligned}
V &= \sum_{i=1}^{2} V_i = V_+ + V_- = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r_+} + \frac{(-q)}{r_-}\right] \\
&= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_+} - \frac{1}{r_-}\right] = \frac{q}{4\pi\varepsilon_0}\left[\frac{r_- - r_+}{r_- r_+}\right]
\end{aligned}
$$

For $r >> d$, $r_+ r_- \approx r^2$ and $(r_- - r_+) \approx d\cos\theta$

$$
\begin{aligned}
V &= \frac{q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2} & p &= qd \\
&= \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} & V(r) &= \text{ scalar field set up by dipole}
\end{aligned}
$$

Azimuthal symmetry (rotational symmetry about z-axis) $\Longrightarrow$ no ϕ dependence. In the equational plane, $\theta = 90^\circ \Longrightarrow V = 0$ because V_+ and V_- cancel ($V_+ = -V_-$).

Continuous Charge Distribution

Given $q(\vec{r'})$, find $V(\vec{r})$.

Recipe for Finding the Potential due to a Continuous Charge Distribution

Method I

1. Divide charge distribution into pieces with charge dq.
2. dq produces a potential $dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r}$ where r is the distance from dq to the point P.
3. $V = \int dV = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r}$. Integrate over the charge distribution. (Notice this assumes $V = 0$ at $r = \infty$)

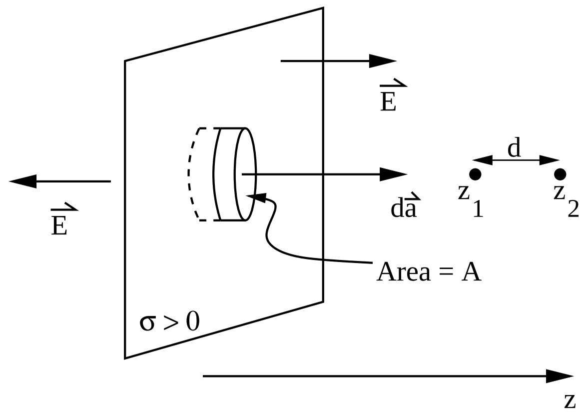
Method II

1. Use Gauss' Law to find $\vec{E}$ if there is symmetry.

2. Use $\Delta V = -\int_a^b \vec{E} \cdot d\vec{s}$

Example Using Method II: Charged Planar Sheet

Consider an infinite (insulating) sheet with uniform positive charge density σ. What is the change in potential in going from z_1 to z_2, where z_1 and z_2 are distances from the sheet?



Solution

Use $V(z_2) - V(z_1) = -\int_{z_1}^{z_2} \vec{E} \cdot d\vec{s}$

1. Find $\vec{E}$ using Gauss' Law:

$$\begin{aligned} \varepsilon_0 \int \vec{E} \cdot d\vec{a} &= q_{\text{enc}} \\ 2\varepsilon_0 E A &= \sigma A \\ E &= \frac{\sigma}{2\varepsilon_0} \end{aligned}$$

Notice that $\vec{E}$ is uniform, i.e., it's the same everywhere.

2. We saw earlier that the potential difference between 2 points in a uniform field is Ed. Since z_2 is further from the sheet of charge than z_1, we expect $V(z_2) - V(z_1) < 0$. So $V(z_2) - V(z_1) = -Ed$. We can also go through the math again:

$$\begin{aligned} V(z_2) - V(z_1) &= -\int_{z_1}^{z_2} \vec{E} \cdot d\vec{s} \\ &= -\int_{z_1}^{z_2} E dz = -E \int_{z_1}^{z_2} dz \\ &= -E(z_2 - z_1) = -\frac{\sigma}{2\varepsilon_0}(z_2 - z_1) \\ V(z_2) - V(z_1) &= -\frac{\sigma}{2\varepsilon_0}(z_2 - z_1) < 0 \end{aligned}$$

Calculating the field from the potential

$$\vec{E} = -\nabla V \tag{9}$$

We've seen how to calculate the potential V if we know $\vec{E}$: $\Delta V = -\int_a^b \vec{E} \cdot d\vec{s}$. How do we calculate $\vec{E}$ if we know V?

$$V = -\int \vec{E} \cdot d\vec{s} \Longrightarrow dV = -\vec{E} \cdot d\vec{s}$$

Suppose $d\vec{s} = dx\ \hat{i}$ ($d\vec{s}$ is in the $\hat{i}$ direction.) Then

$$dV = -\vec{E} \cdot dx\ \hat{i} = -E_x dx$$
$$\Longrightarrow E_x = -\frac{\partial V}{\partial x}$$

If $d\vec{s} = dy\ \hat{j}$, then

$$dV = -E_y dy \to E_y = -\frac{\partial V}{\partial y} \tag{10}$$

If $d\vec{s} = dz\ \hat{k}$, then

$$dV = -E_z dz \Longrightarrow E_z = -\frac{\partial V}{\partial z} \tag{11}$$

In general we can write

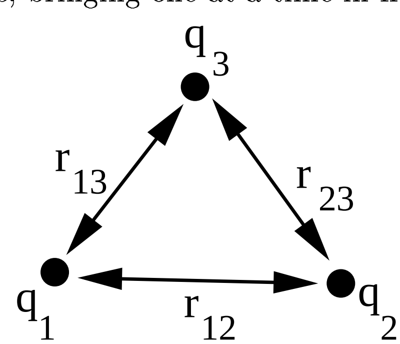
$$\begin{aligned} \vec{E} &= E_x\hat{i} + E_y\hat{j} + E_z\hat{k} \\ &= -\frac{\partial V}{\partial x}\hat{i} = \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial z}\hat{k} \\ &= -\nabla V \end{aligned} \tag{12}$$

where ∇V is called "the gradient of V." What does this mean? Recall the equipotential surfaces, each with constant V. $\vec{E}$ is perpendicular to the surfaces and points from high potential to low potential, i.e., "downhill". So if you put a + charge where $V = 5$ volts, it would feel a force $\vec{F}$ in the direction of $V = 0$ volts. $\vec{F} \parallel \vec{E}$.

Think of a hill. Equipotential corresponds to equi-altitude. $\vec{E}$ would point straight downhill, i.e. along the "fall line".

Electric Potential Energy due to a System of Charges

The electric potential energy of a system of fixed point charges is equal to the work done to assemble the changes, bringing one at a time in from infinity.



Consider assembling 3 point charges. Bring in one charge at a time. No work to bring in q_1, but q_1 sets up a potential

$$V_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r} \tag{13}$$

where r is the distance from q_1. Now bring in q_2. The energy we store in doing this is

$$U_{12} = q_2 V_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}} \tag{14}$$

where r_{12} is the distance between q_2 and q_1. (The work we do is minus the work done by the field, so $U = W_{US} = -W_{\vec{E}\text{ of }q_1}$.)

Note that the general formula for the potential energy of a pair of point charges a distance r apart is

$$U = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}$$

q_2 also sets up a potential $V_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r}$

When we bring in q_3, we add more potential energy.

$$U_{13} + U_{23} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\varepsilon_0}\frac{q_2 q_3}{r_{23}}$$

So the total electric potential energy stored is

$$U_{\text{tot}} = U_{12} + U_{13} + U_{23} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}} + \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\varepsilon_0}\frac{q_2 q_3}{r_{23}}$$

An Isolated Conductor

Recall that $\vec{E} = 0$ inside an isolated conductor. If $\vec{E}$ were not zero, then the free conduction electrons would feel $\vec{F} = q\vec{E}$ and they would run. A charged, isolated conductor has charges on the surface because the charges get as far apart as possible. They arrange themselves such that $\vec{E} = 0$ inside. There is no charge inside because Gauss' law tells us that

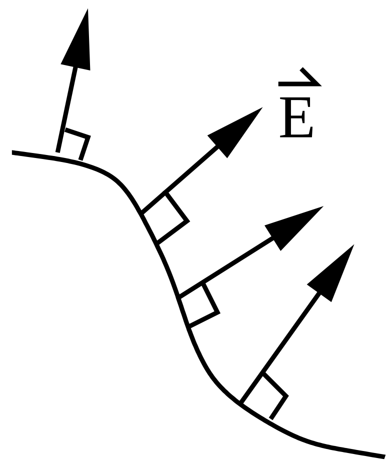
$$\epsilon_0 \oint \vec{E}\cdot d\vec{a} = q_{enc} = 0 \quad \text{because} \quad \vec{E} = 0 \tag{15}$$

V = *Constant throughout a conductor*, i.e., a conductor is an equipotential. Pick 2 points a and b in the conductor or on its surface.

$$V_b - V_a = -\int_a^b \vec{E}\cdot d\vec{s} = 0$$

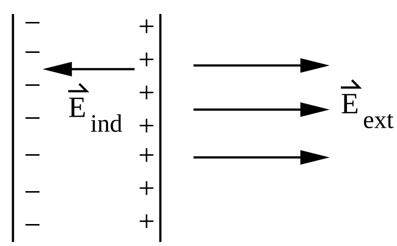
because $\vec{E} = 0$ inside
and $\vec{E} \perp d\vec{s}$ if $d\vec{s}$
is along surface

$$\Longrightarrow \quad V_b = V_a$$



$\vec{E} \perp$ surface of a conductor

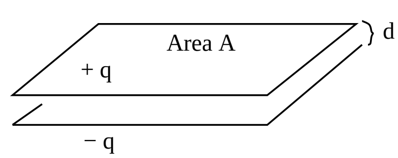
If $\vec{E}$ has a component along the surface, the charges would run along the surface. Here $\vec{E}$ can be either due to charges on the surface of the conductor or due to an external field $\vec{E}_{ext}$. If an uncharged conductor is placed in an external field, the conduction electrons feel the field and race to the surface, arranging themselves until they cancel the field $\vec{E}_{ext}$ inside the conductor. This "induced charge" produces a field $\vec{E}_{ind}$ such that $\vec{E}_{ind} + \vec{E}_{ext} = 0$ inside the conductor.



Capacitors

So far we've been considering a single isolated conductor. Now consider 2 isolated conductors. This is a *capacitor*. The conductors are called *capacitor plates*. Capacitors are important circuit elements. They are also used to store energy.

To understand how they work, consider a parallel plate capacitor consisting of 2 parallel conducting plates a distance d apart. Each plate has area A. When the capacitor is charged, its plates have equal and opposite charges $+q$ and $-q$. Each plate has a constant potential, but there is a potential difference V between the plates. (Conventional notation: $V = \Delta V =$ potential difference between the plates.) V is proportional to q. ("q" is called "the charge of the capacitor," even though both plates are charged.)



$$q = C\Delta V$$

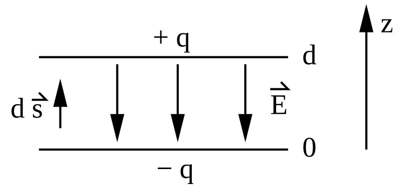
The constant of proportionality C is called the *capacitance* of the capacitor.

Units

$$[C] = \frac{[q]}{[\Delta V]} = \frac{\text{Coulomb}}{\text{volt}} = \text{farad}$$

$$1 \text{ farad} = 1F = \frac{1 \text{ coulomb}}{1 \text{ volt}}$$

1 farad is huge. Typical units: $1\mu F = 10^{-6}F$ and $1pF = 10^{-12}F$. Notice that there is an electric field between the capacitor plates. $\vec{E}$ points from the $+q$ plate to the $-q$ plate.



The $+q$ plate is at a higher voltage than the $-q$ plate. The potential difference between the plates is

$$\begin{aligned}
\Delta V &= -\int_{0(-)}^{d(+)} \vec{E} \cdot d\vec{s} = -\int_{0}^{d} Eds \cos(180°) \\
&= +\int_{0}^{d} E\, ds = \int_{0}^{d} Edz = Ed
\end{aligned}$$

Notice that $\Delta V \propto E$. How do we get $Q = C\Delta V$ from this? Recall Gauss' law:

$$\epsilon_0 \oint \vec{E} \cdot d\vec{a} = q_{enc} \qquad \text{or} \qquad E = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{r^2}$$

This implies $E \propto q$. Since $\Delta V \propto E \propto q$, $\Delta V \propto q$. Thus $\Delta V = \frac{q}{C}$ or $q = C\Delta V$.

Calculating Capacitance

C depends only on the geometry of the conductor, i.e., their size, shape, and separation. C is independent of q and ΔV. In a typical problem, you are given the geometry of the conductors and asked to find the capacitance C. ($C > 0$ always.)

Recipe to find the capacitance

1) Assume $+q$ on one plate and $-q$ on the other plate (or surface charge density $\sigma = q/A$ on one plate, $\sigma = -q/A$ on another plate).

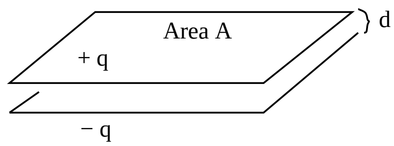
2) Use Gauss' law to find $\vec{E}$ between plates.

3) Use $\Delta V = \int_{(-)}^{(+)} \vec{E} \cdot d\vec{s}$ to find the potential difference between the plates. Note that we want $\Delta V > 0$.

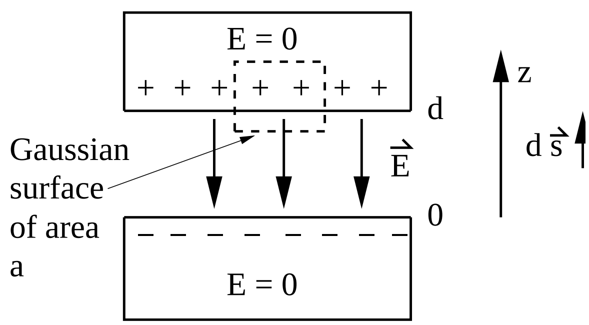
4) Use $q = C\Delta V \Longrightarrow C = \frac{q}{\Delta V}$ to find C. (The charge you put on the plates in step 1 will cancel out.)

Example: Parallel Plate Capacitor

Find the capacitance of a parallel plate capacitor with area A and separation d. ($\sqrt{A} >> d \Longrightarrow$ neglect fringing fields.)



Solution: Put $+q$ on the top plate and $-q$ on the bottom plate. Then the top plate has $\sigma_+ = q/A$ surface charge density and the bottom plate has $\sigma_- = -q/A$. The charge on each plate is attracted to the side facing the other plate. No excess charge is on the outer surfaces. Find $\vec{E}$ using Gauss' law.



$$\varepsilon_o \oint \vec{E} \cdot d\vec{a} = q_{enc} \quad \Longrightarrow \quad \epsilon_0 E a = \sigma_+ a \quad \Longrightarrow E = \frac{\sigma_+}{\epsilon_0} = \frac{q}{\epsilon_0 A} \tag{16}$$

where we used $\sigma_+ = q/A$

Find ΔV using

$$\begin{aligned}\Delta V &= -\int_{(-)}^{(+)} \vec{E}\cdot d\vec{s} = -\int_0^d E\,dz\,\cos(180^\circ) \\ &= +\int_0^d E dz = E\int_0^d dz = Ed \\ &= \frac{q}{\epsilon_0 A} d\end{aligned}$$

where we used $d\vec{s} = dz\,\hat{z}$ and $\vec{E}\cdot d\vec{z} = Edz\cos(180^o)$.

Find

$$C = \frac{q}{\Delta V} = \frac{q}{qd/\epsilon_0 A} = \frac{\epsilon_0 A}{d} \tag{17}$$

Notice that this just depends on the size and separation of the plates, not on q or ΔV. C increases with A and decreases with increasing d. It is more convenient to express

$$\epsilon_0 = 8.85\times 10^{-12}\frac{\mathrm{F}}{\mathrm{m}} = 8.85\ \mathrm{pF/m}$$

(Before we used different units: $\epsilon_0 = 8.85\times 10^{-12}\ \mathrm{C^2/(N\text{-}m^2)}$).

If the plates are 1 mm apart ($d = 10^{-3}m$) and $C = 1F$, how big are the plates?

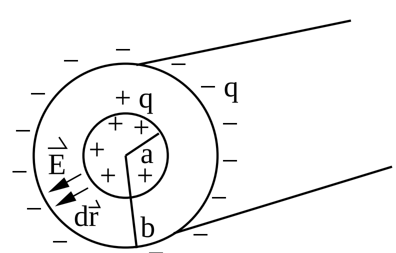
$$A = \frac{d}{\epsilon_0}C = \frac{(10^{-3}\mathrm{m})(1\mathrm{F})}{8.85\times 10^{-12}\mathrm{F/m}} = 1.13\cdot 10^8\mathrm{m}^2$$

If the plates are square, then each side is

$$\sqrt{A} = 1.0\times 10^4\mathrm{m} = 10\ \mathrm{km} \qquad \text{huge!}$$

Example: Cylindrical Capacitor

Consider 2 coaxial cylinders of length L and radii a and b. Find the capacitance C.



Solution: Assume the inner cylinder has charge $+q$ and $\sigma_+ = q/A_+ = q/2\pi aL$. The outer cylinder has $-q$ and $\sigma_- = -q/A_- = q/2\pi bL$.
Find $\vec{E}$ using Gauss' law. The Gaussian surface is a cylinder of length $\ell < L, a < r < b$.

$$\begin{aligned}
\epsilon_0 \oint \vec{E} \cdot d\vec{a} &= q_{enc} \Longrightarrow \epsilon_0 E(2\pi r\ell) = \sigma_+ 2\pi a\ell \\
\vec{E} &= \frac{1}{\epsilon_0}\frac{\sigma_+ a}{r}\hat{r}
\end{aligned}$$

Find ΔV:

$$\Delta V = -\int_{(-)}^{(+)} \vec{E} \cdot d\vec{s} \quad \Longrightarrow \quad -\Delta V = -\int_{(+)}^{(-)} \vec{E} \cdot d\vec{s}$$

where we switched limits so that $\vec{E}||d\vec{r}||d\vec{s}$.

$$\begin{aligned}
-\Delta V &= -\int_{(+)}^{(-)} \vec{E} \cdot d\vec{s} = -\int_{(+)}^{(-)} E\,dr \qquad\qquad \text{use } d\vec{s} = d\vec{r} \text{ and } \vec{E}||d\vec{r} \\
&= -\int_{r=a}^{r=b} \frac{1}{\varepsilon_o}\frac{\sigma_+ a}{r} dr \\
&= -\frac{\sigma_+ a}{\epsilon_0} \ln\left(\frac{b}{a}\right) \\
\Delta V &= +\frac{\sigma_+ a}{\epsilon_0} \ln\left(\frac{a}{b}\right) = \frac{qa}{2\pi\epsilon_0 aL} \ln\left(\frac{b}{a}\right) = \frac{q}{2\pi\epsilon_0 L} \ln\left(\frac{b}{a}\right)
\end{aligned}$$

Find C:

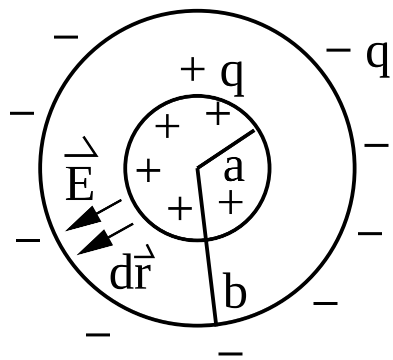
$$q = C\Delta V \quad \Longrightarrow \quad C = \frac{q}{\Delta V} \tag{18}$$

$$C = \frac{q}{\Delta V} = \frac{q}{\frac{q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right)} = 2\pi\epsilon_0 \frac{L}{\ln\left(\frac{b}{a}\right)}$$

Again C only depends on geometrical quantities.

Spherical Capacitor

A spherical capacitor consists of 2 concentric spherical conducting shells of radii a and b. Find the capacitance C.



Solution: Assume the inner sphere has charge $+q$. Find $\vec{E}$ between the spheres. We know that outside the sphere of charge $+q$, $\vec{E}$ is the same as for a point charge:

$$\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r} \tag{19}$$

We could also use Gauss' law: $\epsilon_0 \oint \vec{E} \cdot d\vec{a} = q_{enc}$

$$\epsilon_0 E \cdot 4\pi r^2 = q \quad \Longrightarrow \quad E = \frac{q}{4\pi\epsilon_0 r^2} \tag{20}$$

Find the potential difference between the spheres

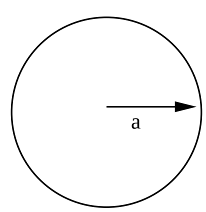
$$\Delta V = -\int_{(-)}^{(+)} \vec{E} \cdot d\vec{s} \quad \Longrightarrow \quad -\Delta V = -\int_{(+)}^{(-)} \vec{E} \cdot d\vec{s} \qquad d\vec{s} = d\vec{r} || \vec{E}$$

$$\begin{aligned} -V &= -\int_{(+)}^{(-)} \vec{E} \cdot d\vec{s} = -\int_{r=a}^{r=b} E\,dr = -\frac{q}{4\pi\epsilon_0}\int_{r=a}^{r=b} \frac{dr}{r^2} \\ &= -\frac{q}{4\pi\epsilon_0}\left[-\frac{1}{r}\right]_{r=a}^{r=b} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{b} - \frac{1}{a}\right] \\ \Longrightarrow +V &= -\frac{q}{4\pi\epsilon_0}\left[\frac{1}{b} - \frac{1}{a}\right] = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right] \end{aligned}$$

$$C = \frac{q}{\Delta V} = \frac{q}{\frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)} = \frac{4\pi\epsilon_0}{\frac{1}{a} - \frac{1}{b}}$$

Note that C just depends on geometry

Suppose we have an isolated conductor of radius a.



We can assign it a capacitance by assuming the other (outer) sphere is at infinity. So if we set $b = \infty$, then

$$C = \frac{4\pi\epsilon_0}{1/a} = 4\pi\epsilon_0 a$$

Numbers for Van de Graaf

$a \cong 10$ cm $= 0.2$ m
$\epsilon_0 = 8.85$ pF/m

$$\begin{aligned}
C &= 4\pi\epsilon_0 a = 4\pi\left(8.85\,\frac{\mathrm{pF}}{\mathrm{m}}\right)(0.1\mathrm{m}) = 11\mathrm{pF} \\
Q &= C\Delta V \qquad \Delta V = Ed \qquad d = \text{distance spark jumps} = 3\text{ cm} = 0.03\text{ m} \\
E &= 3\times 10^6 V/m \qquad \text{breakdown of air}
\end{aligned}$$

$$\begin{aligned}
\Delta V &= Ed = \left(3\times 10^6 V/m\right)(0.03m) = 9\times 10^4\ V = 90,000V \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{High voltage!} \\
Q &= C\Delta V = (11\ pF)(90,000V) = (11\times 10^{-12}F)(90,000V) \\
&= 9.9\times 10^{-7}C \\
&\simeq 10^{-6}C \qquad\qquad \text{small amount of charge}
\end{aligned}$$

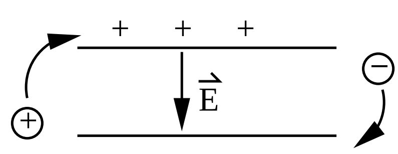
Energy released

$$U = \frac{1}{2}C(\Delta V)^2 = \frac{1}{2}(11\times 10^{-12}F)(90,000V)^2 = 4.5\times 10^{-2}\ \text{ Joules} \tag{21}$$

If this is released in 1 second, power $= U/\Delta t = 0.045$ Watts.

Storing Energy in the $\vec{E}$ field

We have seen that capacitors store charge. It turns out that charging capacitors is also a way to store energy.



To charge up a capacitor, you have to remove electrons from the positive plate and carry them to the negative plate. In doing so, you fight against the electric field, which is pulling them back toward the positive plate and repelling them from the negative plate. How much work does this take? If a voltage ΔV has already built up, $\Delta V = q/C$. The work done in transferring charge dq (imagine $dq > 0$ being transferred from the negatively charged plate to the positively charged plate) is

$$dW = \Delta V\, dq = \frac{q}{C}dq$$

The total work in going from $q = 0$ to $q = Q$ is

$$W = \int dW = \int_0^Q \frac{q}{C} dq = \frac{1}{2}\frac{Q^2}{C}$$

This work is stored as potential energy in the capacitor

$$\boxed{U = \frac{1}{2}\frac{Q^2}{C}}$$

Since $Q = C\Delta V$, $U = \frac{1}{2}C(\Delta V)^2$. So

$$\boxed{U = \frac{1}{2}C(\Delta V)^2}$$

The bigger the voltage or charge, the more the energy that is stored. An example of energy stored in a capacitor is a camera flash.

Where is the energy residing? It is viewed as being stored in the electric field $\vec{E}$. It is customary to define an *energy density* u as the potential energy per unit volume (of the space with E). Consider a parallel plate capacitor with voltage difference ΔV and a uniform electric field $\vec{E} = \frac{\Delta V}{d}(-\hat{z})$. Using $C = \varepsilon_0 A/d$, we can write

$$\begin{aligned} u &= \frac{U}{\text{volume}} = \frac{U}{Ad} = \frac{1}{2}\frac{C(\Delta V)^2}{Ad} = \frac{\epsilon_0}{2}\left(\frac{\Delta V}{d}\right)^2 \\ \Delta V &= \int \vec{E} \cdot d\vec{s} \quad \Longrightarrow \quad E = \frac{\Delta V}{d} \quad \Longrightarrow \quad \boxed{u = \epsilon_0 E^2/2} \end{aligned}$$

Although we derived this result for the special case of a parallel plate capacitor, it holds in general. (Just imagine the field being uniform in a tiny element of space.) This formula is true even if there isn't a capacitor. If $E \neq 0$, $u = \epsilon_0 E^2/2$. Note that superposition does not hold:

$$\begin{aligned} \vec{E} = \vec{E}_1 + \vec{E}_2 \Longrightarrow \quad u &= \frac{\epsilon_0}{2}E^2 = \frac{\epsilon_0}{2}(\vec{E}_1 + \vec{E}_2)^2 \\ u &= \frac{\epsilon}{2}(\vec{E}_1^2 + \vec{E}_2^2 + 2\vec{E}_1 \cdot \vec{E}_2) \\ &\neq \frac{\epsilon}{2}\vec{E}_1^2 + \vec{E}_2^2 \end{aligned}$$

So add fields and then square to get u.

Capacitor with Dielectric

It is common to find capacitors that are $1F$ but aren't 10 km on a side. How do we increase C without increasing the dimensions of the capacitor?

Answer: We can place an insulator, such as plastic, glass, or oil, between the capacitor plates. This insulator is called a *dielectric*. It increases the capacitance by a numerical factor κ called the *dielectric constant*. Thus, if we place a dielectric with dielectric constant κ between the plates of a capacitor, the capacitance increases:

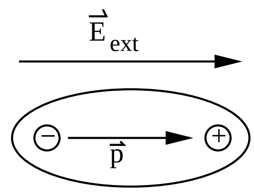
$$C = \kappa C_0 \tag{22}$$

where C_0 is the capacitance without the dielectric. For a parallel place capacitor,

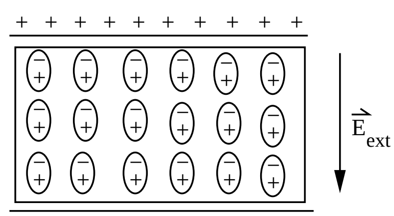
$$C_0 = \epsilon_0 \frac{A}{d} \quad \Longrightarrow \quad C = \kappa C_0 = \kappa \epsilon_0 \frac{A}{d}$$

$\kappa = 1$ for vacuum. $\kappa \geq 1$ for all other materials. So $C \geq C_0$, i.e., the capacitance increases.

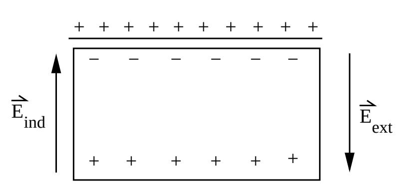
Why does this happen? Microscopically the molecules in the dielectric become polarized by the electric field between the plates of a charged capacitor. "Polarized" means that the electric dipole moments of the molecules line up in the field. These dipole moments are either "permanent" dipole moments (like H_20) or "induced" dipole moments (or both, i.e., permanent moments enhanced by induction). An induced dipole moment occurs when the external $\vec{E}$ field "stretches" a molecule by separating the positive and negative charges a little. This creates



Inside the dielectric there is no excess charge. On the surface near the positively charged plate, the dielectric has a build up of negative charge. Similarly, the negatively charged plate attracts a buildup of positive charge in the dielectric. The dielectric is neutral overall, i.e., no net charge, but it is polarized. The dielectric charges "screen" the charges on the plate, making them less repulsive to each other and more willing to stay.



For example, the positive charge on the upper plate is attracted to the negative charge on the surface of the dielectric, so it is more willing to stay than before the dielectric was there. (This is like feeling more comfortable at a party if you know some people there.) Thus the dielectric enables us to put more charge on the capacitor plates, i.e., to increase the capacitance.



Notice that the induced field $\vec{E}_{ind}$ in the dielectric is opposite to $\vec{E}_{ext}$. As a result the total field $\vec{E}_{tot}$ between the plates is reduced.

$$\vec{E}_{tot} = \vec{E}_{ext} + \vec{E}_{ind}, \qquad |\vec{E}_{tot}| < |\vec{E}_{ext}|$$

For a given amount of (free) charge Q on the plates, ΔV is reduced when the dielectric is present. ("Free" charge refers to the charge that is not induced.)

$$\Delta V = -\int \vec{E}_{tot} \cdot d\vec{s} \tag{23}$$

$$\underset{\Delta V_0 = Q/C}{\underline{\text{empty}}} \qquad \Longrightarrow \Delta V_\kappa < \Delta V_0 \qquad \underset{\Delta V_\kappa = Q/\kappa C_0}{\underline{\text{with dielectric}}}$$

where Q is the free charge and $\kappa > 1$. To achieve a given voltage ΔV between the plates

$$Q_0 = C_0 \Delta V \qquad \Longrightarrow Q_\kappa > Q_0 \qquad Q_\kappa = \kappa C_0 \Delta V$$

Therefore a capacitor can hold more free charge with the dielectric. Charging the capacitor to voltage ΔV stores energy U

$$U_0 = \frac{1}{2}\frac{Q_0^2}{C_0} = \frac{1}{2}C_0(\Delta V)^2 \quad < \quad U_\kappa = \frac{1}{2}\frac{Q_\kappa^2}{\kappa C_0} = \frac{1}{2}\kappa C_0 (\Delta V)^2$$

More charge must be pumped onto the dielectric capacitor to achieve a given voltage ΔV. This takes more work. So more energy U_κ is stored.

Putting free charge Q on the capacitor stores energy U

$$U_0 = \frac{1}{2}C_0 \Delta V_0^2 = \frac{1}{2}\frac{Q^2}{C_0} > U_\kappa = \frac{1}{2}\kappa C_0 (\Delta V)_\kappa^2 = \frac{1}{2}\frac{Q^2}{\kappa C_0}$$

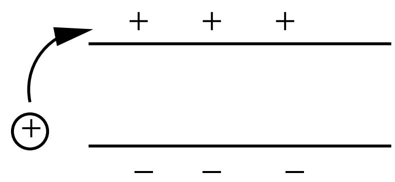
It takes less work to put Q on the dielectric capacitor because the charge is screened.

Gauss' law when a dielectric is present:

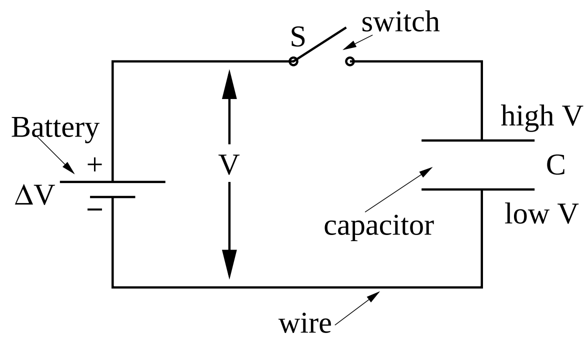
$$\epsilon_0 \oint \kappa \vec{E} \cdot d\vec{a} = q_{\text{free}}$$

Capacitors in Circuits

Capacitors are important circuit elements. One way to charge a capacitor is to attach it to a battery. You can think of a battery as a pump which takes plus charge from the negatively charged plate and adds it to the positively charged plate.



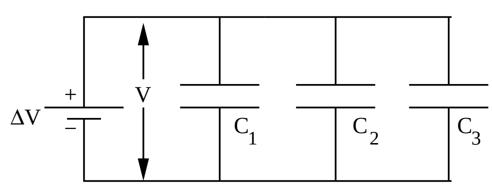
The battery keeps doing this until the voltage across the terminals of the battery equals the voltage across the capacitor plates. Let's draw a circuit diagram to show the setup for doing this. An electric circuit is the path through which current flows.



Current starts to flow and the capacitor starts charging once the switch S is closed. Once C is charged, it will stay charged even if S is opened, i.e., $+q$ and $-q$ stay on the plates.

Sometimes we have more than one capacitor in a circuit. It is convenient to replace combinations of capacitors with an equivalent capacitor C_{eq}. This simplifies the circuit. There are two basic combinations.

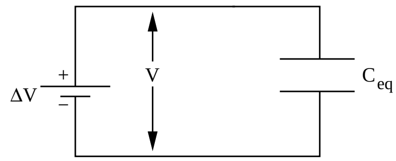
Capacitors in Parallel



Capacitors are connected in parallel when a potential difference applied across their combination results in that potential difference being applied across each capacitor. The equivalent capacitor has the same amount of charge as all the parallel capacitors combined and the same potential drop as all capacitors combined. In the case shown

$$\begin{aligned} q_1 &= C_1\Delta V \qquad q_2 = C_2\Delta V \qquad q_3 = C_3\Delta V \\ \text{total charge} &= q = q_1 + q_2 + q_3 = C_1\Delta V + C_2\Delta V + C_3\Delta V \\ &= (C_1 + C_2 + C_3)\Delta V \\ &= C_{eq}\Delta V \quad \text{where} \quad C_{eq} = C_1 + C_2 + C_3 \end{aligned}$$

C_{eq} is the equivalent capacitance of the single capacitor.

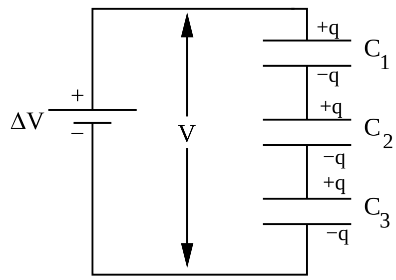


In general,

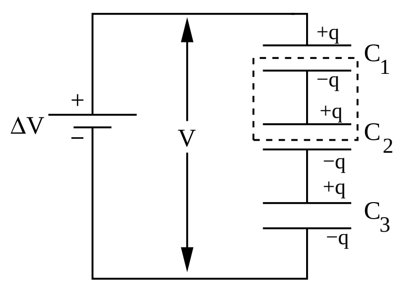
$$C_{eq} = \sum_{i=1}^{n} C_i \qquad \text{for capacitance in parallel}$$

Add the capacitances for capacitors in parallel.

Capacitors in Series



Capacitors are connected in series when the potential difference applied across the combination is the sum of the potential differences across each capacitor. C_{eq} has the same q and the same ΔV as the whole combination.



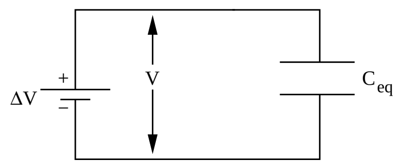
Notice that each capacitor in the series has the same amount of charge on it. To see this, note that the boxed conductor is electrically isolated and therefore has no net charge. The charges are merely separated into $+q$ and $-q$. No charge can be transferred to the isolated element.

$$\begin{aligned} q &= C_1 \Delta V_1 \qquad & q = C_2 \Delta V_2 \qquad & q = C_3 \Delta V_3 \\ \Delta V_1 &= \frac{q}{c_1} \qquad & \Delta V_2 = \frac{q}{C_2} \qquad & \Delta V_3 = \frac{q}{C_3} \end{aligned}$$

By the definition of being in series,

$$\begin{aligned}
\Delta V &= \Delta V_1 + \Delta V_2 + \Delta V_3 = \frac{q}{C_1} + \frac{q}{C_2} + \frac{q}{C_3} \\
&= q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right) \\
&= q\frac{1}{C_{eq}} \\
\frac{1}{C_{eq}} &= \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}
\end{aligned}$$

$$C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}} \tag{24}$$



In general, for n capacitors in series

$$\frac{1}{C_{eq}} = \sum_{i=1}^{n} \frac{1}{C_i} \tag{25}$$