Lectures on Chapter 19: Electric Forces and Electric Fields

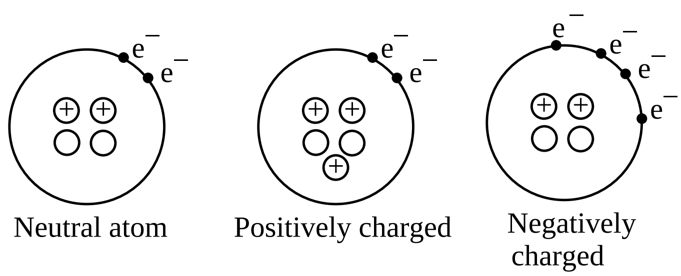
The modern world as we know it would not be possible without electricity and magnetism. Television, computers, circuits, microwaves, etc. all rely on the principles of electricity and magnetism. I doubt if the scientists working to understand these principles in the 1700's and 1800's ever dreamed of the technology we have today.

Charge

Charge is the basic entity. There are 2 kinds of charge: positive and negative. Like charges repel and unlike charges attract:

$\leftarrow\ominus\ \ominus\rightarrow \qquad \leftarrow\oplus\ \oplus\rightarrow \qquad \oplus\rightarrow\leftarrow\ominus \qquad$ Never get $\ominus\rightarrow\ \ \ominus\rightarrow$

The origin of these charges are electrons which are negatively charged and protons which are positively charged. Atoms consist of electrons, protons, and neutrons. Neutrons are electrically neutral. The nucleus has protons and neutrons. The electrons orbit around it.



If the number of protons equals the number of electrons ($N_p = N_e$), the atom is neutral. If $N_p > N_e$, the atom is positively charged. If $N_e > N_p$, the atom is negatively charged.

A simple way to "charge" an object is to rub it. For example if you rub a glass or plastic rod with silk, some of the electrons get rubbed off onto the silk. As a result the rod becomes positively charged and the silk becomes negatively charged. Another example is combing your hair. Now there are more protons than electrons in your hair, so it's positively charged. The comb, on the other hand, has more electrons and is negatively charged. You can then use the charged comb to pick up bits of paper which are slightly positively charged. (Actually the molecules in the paper are polarized by the charged comb. They are polarized in such a way as to be attracted to the comb. More on this later.)

Things don't like to be charged. They like to be neutral. The negatively charged comb attracts positively charged objects. The electrons from the comb want to jump to the positively charged hair. Once the hair and comb are neutral, there is no longer any desire to transfer electrons and there is no attraction. Jumping charges create sparks and lightning.

Charge is conserved: That means that charge doesn't spontaneously appear or disappear. Electrons can go from your hair to the comb. But if you lose 4 electrons from your hair, your comb gains 4 electrons.

Units: The MKS unit of charge is the Coulomb. One Coulomb (1C) is a *lot* of charge. A lightning bolt consists of 20C of charge flowing from sky to earth. An electron has a tiny amount of charge: $1e = 1.60 \times 10^{-19}C$. All electrons are the same; each has $1e$ of negative charge. Protons have $1e$ of positive charge. We say that charge is "quantized" because it comes in these discrete packets. You can find objects with $q = 1e, 2, e$, or $10e$, but never non-integer amounts of charge like $3.5e$.

Conductors and Insulators

You can't charge everything by rubbing it. For example, you can't charge a copper rod by rubbing it because copper is not good at holding charge. Copper is a conductor. Charge runs freely through conductors. Your body is a good conductor and so is water because they have ions. So if you hold a copper rod and rub it, the excess charge runs through the rod, through you, and onto the floor. As a result, the rod doesn't charge up. (The fact that you are a good conductor means that you can be electrocuted.) Metals are generally good conductors because electrons flow freely through them. These are called "conduction electrons." This is why electrical wire is made of metal like copper. If you put a bunch of charge on an isolated conductor, e.g. a bunch of negative charge, it won't pile up in one place. Since electrons repel each other, they will get as far from each other as possible. So for example, the charge will spread uniformly over the surface of a spherical conductor.

The opposite of conductors are *insulators*. Charge does not flow in insulators; it's stuck. Rubber and glass are examples of insulators. In these materials the electrons are stuck in covalent bands. You can charge insulators by rubbing them because they are good at "holding charge." For example, you can charge a glass rod or a plastic comb by rubbing it.

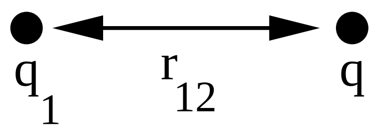
Charging by Induction

There is a way to charge objects without rubbing them. Namely, one can charge objects by inducing charge on them. Suppose you take a positively charged rod and bring it up to a neutral object. Let's suppose that the object is a conductor. Charge is free to run on a conductor, so negative charges in the conductor would be attracted to the rod and positive charges in the conductor would be repelled away from the rod. This separation of positive and negative charge is called **polarization**. We say that the rod has polarized the conductor. Suppose we ground the conductor. By that, I mean suppose we connect the conductor by a wire to an infinite reservoir of charge so that the positive charge that is repelled by the rod can flow into the reservoir, leaving negative charge behind on conductor. If we disconnect the grounding wire, the conductor has a net negative charge and we have charged the conductor by induction. Notice that we didn't have to touch the conductor in order to charge it.

Now suppose that we bring the positively charged rod near an insulator. Charge is not free to roam in an insulator but we can distort the charge distribution and polarize the insulator. Microscopically the electron cloud in an insulator molecule near the rod will be attracted toward the rod, leaving a deficit of negative charge in the part of the molecule that is further away from the rod. This deficit of negative charge will be positively charged. So the molecule will have more positive charge at one end than the other. We say that the rod has polarized the molecule. The induced charge distribution makes the molecule attracted toward the rod. This is how van der Waals interactions work. A polarized molecule induces other molecules to be polarized in such a way as to be attracted to the initially polarized molecule. This polarization explains why a comb that has been rubbed against your hair can attract bits of neutral paper.

Coulomb's Law

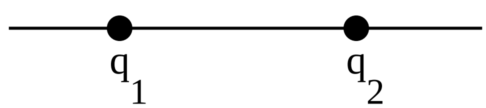
Let q denote the amount of charge, e.g., $3e$. The electrostatic force between charge q_1 and q_2 a distance r_{12} apart



is given by Coulomb's law

$$\vec{F}_{1\leftarrow 2} = k_e \frac{q_1 q_2}{r_{12}^2}\hat{r}_{12} \qquad \begin{array}{l} k_e = \text{constant} \\ \hat{r}_{12} = \frac{\vec{r}_1 - \vec{r}_2}{|r_1 - r_2|} \end{array}$$

The force is a vector. In which direction does it point? It points along the line you can draw through the two point charges.



If q_1 and q_2 are both positive or both negative,

$$q_1 q_2 > 0 \quad \Longrightarrow F > 0 \quad \Longrightarrow \text{ repulsion } \overset{\longleftarrow\!\bullet}{q_1} \quad \overset{\bullet\!\longrightarrow}{q_2}$$

If q_1 and q_2 are oppositely charged,

$$q_1 q_2 < 0 \quad \Longrightarrow F < 0 \quad \Longrightarrow \quad \text{attraction } \overset{\bullet\!\longrightarrow}{q_1} \quad \overset{\longleftarrow\!\bullet}{q_2}$$

Notice that the magnitude of the force diminishes rapidly as the charges get farther apart: $F \propto \frac{1}{r^2}$. The force falls off as $1/r^2$. Thus if $F = 1N$ when $r = 1m$, then $F = \frac{1}{4}N$ when $r = 2m$, and $F = \frac{1}{9}N$ when $r = 3m$.

The constant $k_e = \frac{1}{4\pi\epsilon_0}$, where $\pi = 3.14\cdots$ and is dimensionless. ϵ_0 is called the permittivity constant and it has dimensions.

$$\begin{aligned} \epsilon_0 &= 8.85 \times 10^{-12}\,\mathrm{C^2/N - m^2} \\ k_e = \frac{1}{4\pi\epsilon_0} &= 8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2} \end{aligned}$$

Notice that Coulomb's law has the same form as the gravitational force law:

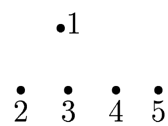
$$F = G\frac{m_1 m_2}{r^2}$$

We get Coulomb's law if we let $m \longrightarrow q$ and $G \longrightarrow \frac{1}{4\pi\epsilon_0}$. The electrostatic force is a billion, billion, billion, billion times stronger than the gravitational force. Consider an electron and a proton. The ratio of the electrostatic force to the gravitational force between them is

$$\frac{F_e}{F_g} = \frac{\frac{1}{4\pi\epsilon_0} q_e q_p}{G\, m_e m_p} = 2 \cdot 10^{39} \quad (\sim 10^{42} \text{ for 2 electrons})$$

Principle of Superposition

Coulomb's law tells us the force that charge q_2 exerts on q_1. What if we have more than 2 charges? Suppose we have 5 charges.



What is the force on q_1? The principle of superposition tells us that we can add the force of each charge on q_1 to get the total force:

$$\vec{F}_{1tot} = \vec{F}_{1\leftarrow 2} + \vec{F}_{1\leftarrow 3} + \vec{F}_{1\leftarrow 4} + \vec{F}_{1\leftarrow 5}$$

$\vec{F}_{1\leftarrow 2}$ means the force on q_1 due to q_2. Notice that there is no $\vec{F}_{1\leftarrow 1}$ because q_1 does not exert a direct force on itself. If it did, it would be infinite:

$$\vec{F}_{1\leftarrow 1} = k_e \frac{q_1^2}{r^2} \longrightarrow \infty \quad \text{as} \quad r \rightarrow 0.$$

So point charges don't exert forces on themselves.

Direction of the force in Coulomb's Law

We have said that charges q_1 and q_2 feel a force whose magnitude is given by Coulomb's law.

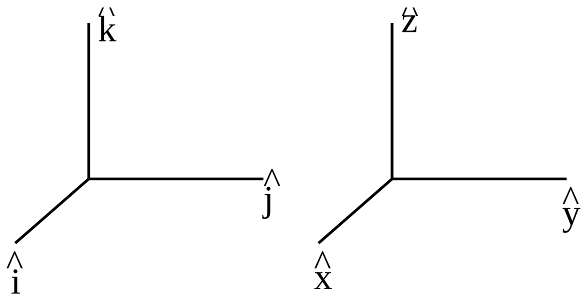
$$F = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2}$$

But force is a vector and it has a direction. It points along the line we can draw through the two points. How do we describe this mathematically? Just saying "That way" isn't good enough. So we want the vector components of $\vec{F}$:

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the x, y, and z-axes. Sometimes I use $\hat{x}, \hat{y}, \hat{z}$ instead.

$$\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix}$$

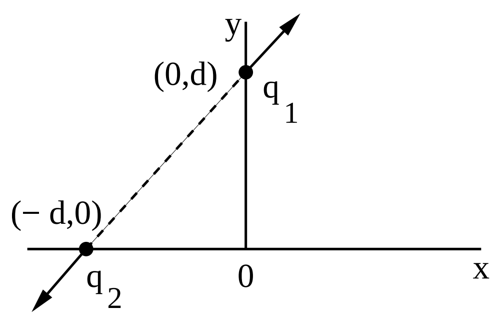


Notice that the force $\vec{F}_{1\leftarrow 2}$ on q_1 due to q_2 is equal and opposite to the force $F_{2\leftarrow 1}$ on q_2 due to q_1.

$$\begin{array}{ccc} F_{1\leftarrow 2} & & F_{2\leftarrow 1} \\ \oplus \longrightarrow & & \longleftarrow \ominus \\ q_1 & & q_2 \end{array} \qquad \begin{array}{ccc} F_{1\leftarrow 2} & & F_{2\rightarrow 1} \\ \longleftarrow \oplus & & \oplus \longrightarrow \\ q_1 & & q_2 \end{array}$$

attractive   repulsive

So we need to specify $\vec{F}_{1\leftarrow 2}$ or $\vec{F}_{2\leftarrow 1}$ to know which vector, and hence which direction, we want. There are two ways to get the components of $\vec{F}$: (1) trigonometry and (2) vector components. It's easier to explain this with an example.



Example: Suppose we have 2 point charges q_1 and $+e$ located at $\vec{r}_1 = (0, d)$ and $q_2 = +2e$ located at $\vec{r}_2 = (-d, 0)$. What is the force felt by q_1?

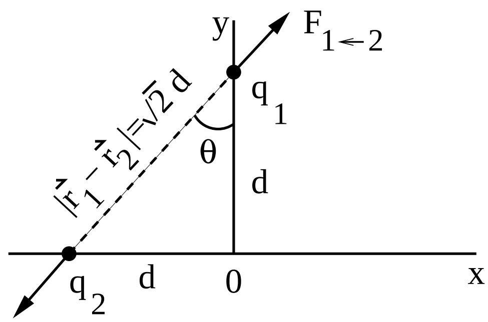
Is the force attractive or repulsive? It's repulsive.

Magnitude: First calculate the magnitude of the force

$$F_{1\leftarrow 2} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{(\vec{r}_1 - \vec{r}_2)^2}$$

(absolute value sign in $|F_{1\leftarrow 2}|$ gets rid of $q_1 q_2 \gtrless 0$, so don't use it.)

$$
\begin{aligned}
(\vec{r}_1 - \vec{r}_2)^2 &= (r_{1x} - r_{2x})^2 + (r_{1y} - r_{2y})^2 \\
&= (0 - (-d))^2 + (d - 0)^2 \\
&= d^2 + d^2 \\
&= 2d^2 \qquad \Longrightarrow \qquad r_{12} = |\vec{r}_1 - \vec{r}_2| = \sqrt{2}d \\
F_{1\leftarrow 2} &= \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{(\vec{r}_1 - \vec{r}_2)^2} = \frac{1}{4\pi\epsilon_0}\frac{\not{2}e^2}{\not{2}d^2} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{d^2}
\end{aligned}
$$



Direction
Method I: Trigonometry

$$
\begin{aligned}
&\vec{F}_{1\leftarrow 2} = F_{1\leftarrow 2}\sin\theta\,\hat{\imath} + F_{1\leftarrow 2}\cos\theta\,\hat{\jmath} \\
&\sin\theta = \frac{d}{r_{12}} = \frac{d}{\sqrt{2}d} = \frac{1}{\sqrt{2}} \\
&\cos\theta = \frac{d}{r_{12}} = \frac{1}{\sqrt{2}} \\
&\vec{F}_{1\leftarrow 2} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{d^2}\left\{\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}\right\}
\end{aligned}
$$

Method II: Vectors:

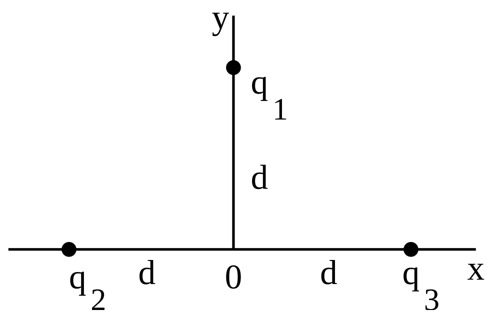
$$
\begin{aligned}
&\vec{F}_{1\leftarrow 2} = F_{1\leftarrow 2}\hat{r}_{12} = \text{magnitude} \cdot \text{direction} \\
&\hat{r}_{12} = \text{unit vector} = \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|} \qquad \longleftarrow \text{divide by magnitude to just get direction} \\
&\qquad = \frac{\vec{r}_1 - \vec{r}_2}{r_{12}} = \frac{\vec{r}_1 - \vec{r}_2}{\sqrt{2}d} \\
&\qquad = \frac{(r_{1x} - r_{2x})\hat{\imath} + (r_{1y} - r_{2y})\hat{\jmath}}{\sqrt{2}d} \qquad\qquad \begin{array}{l}\vec{r}_1 = (0, d) \\ \vec{r}_2 = (-d, 0)\end{array}
\end{aligned}
$$

$$= \frac{(0-(-d))\hat{\imath} + (d-0)\hat{\jmath}}{\sqrt{2}d}$$
$$= \frac{\cancel{d}\hat{\imath} + \cancel{d}\hat{\jmath}}{\sqrt{2}\ \cancel{d}}$$
$$= \frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath})$$

Notice that $\hat{r}_{12} \cdot \hat{r}_{12} = 1$ as it should for a unit vector.

$$\vec{F}_{1\leftarrow 2} = F_{1\leftarrow 2}\hat{r}_{12} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{d^2}\frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath})$$

Now add a 3rd charge $q_3 = q_2 = +2e$ at $(d, 0)$. What is the force on q_1?



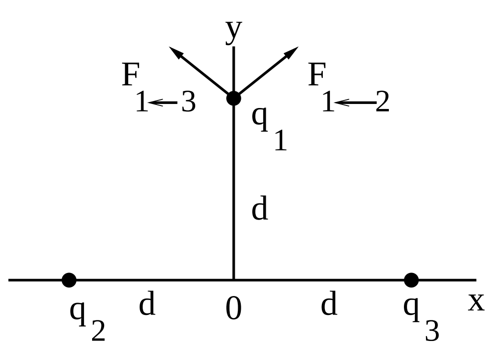
The principle of superposition tells us

$$\vec{F}_{1,tot} = \vec{F}_{1\leftarrow 2} + \vec{F}_{1\leftarrow 3}$$

A common mistake is to add the magnitudes:

$$F_{1,total} = F_{1\leftarrow 2} + F_{1\leftarrow 3} \longleftarrow \underline{\text{wrong}}$$

$$F_{1,total} \neq \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2} + \frac{1}{4\pi\epsilon_0}\frac{q_1 q_3}{r_{13}^2}$$



We must add the *vectors*. Let's look for symmetry. It's always good to look for symmetry because it can save you a lot of work. By symmetry, there are equal and opposite forces in the x direction that cancel out. So the net total force $F_{1,total}$ is parallel to $+\hat{\jmath}$. So we only have to add the y-components of $\vec{F}$.

$$\begin{aligned}\vec{F}_{1,total} &= \left[\left(\vec{F}_{1\leftarrow 2}\right)_y + \left(\vec{F}_{1\leftarrow 3}\right)_y\right]\hat{\jmath} \\ &= \left[F_{1\leftarrow 2}\cos\theta + F_{1\leftarrow 3}\cos\theta\right]\hat{\jmath} \\ &= \left[\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2}\cos\theta + \frac{1}{4\pi\epsilon_0}\frac{q_1 q_3}{r_{13}^2}\cos\theta\right]\hat{\jmath}\end{aligned}$$

Use $r_{12}^2 = 2d^2 = r_{13}^2$ and $\cos\theta = \frac{1}{\sqrt{2}}$

$$\begin{aligned}\vec{F}_{1,total} &= \frac{1}{4\pi\epsilon_0}\frac{\cancel{2}e^2}{\cancel{2}d^2}\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right]\hat{\jmath} \\ &= \frac{\sqrt{2}}{4\pi\epsilon_0}\frac{e^2}{d^2}\hat{\jmath}\end{aligned}$$

We could also just calculate $\vec{F}_{1,total}$ without noticing the symmetry. We plug into

$$\vec{F}_{1,total} = \vec{F}_{1\leftarrow 2} + \vec{F}_{1\leftarrow 3}$$

We already know $F_{1\leftarrow 2} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{d^2}\frac{1}{\sqrt{2}}(\hat{\imath}+\hat{\jmath})$. Going through the same steps as for $\vec{F}_{1\leftarrow 2}$, we get

$$\begin{aligned}\vec{F}_{1\leftarrow 3} &= \frac{1}{4\pi\epsilon_0}\frac{e^2}{d^2}\frac{1}{\sqrt{2}}(-\hat{\imath}+\hat{\jmath}) \\ \vec{F}_{1,total} &= \vec{F}_{1\leftarrow 2} + \vec{F}_{1\leftarrow 3} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{d^2}\frac{1}{\sqrt{2}}\left[(\hat{\imath}+\hat{\jmath}) + (-\hat{\imath}+\hat{\jmath})\right] \\ &= \frac{1}{4\pi\epsilon_0}\frac{e^2}{d^2}\frac{2}{\sqrt{2}}\hat{\jmath} = \frac{\sqrt{2}}{4\pi\epsilon_0}\frac{e^2}{d^2}\hat{\jmath} \quad \text{same as before}\end{aligned}$$

Electric Field

Coulomb's law tells us that a charge q_1 exerts a force on q_2 a distance r away. How does it exert a force without even touching q_2? We "explain" this action-at-a-distance by saying that q_1 sets up an **electric field** around itself.

What do we mean by a field?

A field is any physical quantity that takes on different values at different points in space (and maybe even time). Think of a topographic map that shows a terrain. Each point (x, y) is associated with a height $h(x, y)$ above sea level. This is called a **scalar field** since only one number is associated with (x, y). A **vector field** has a vector associated with each point in space. For example, consider a river. At each point you can assign a velocity $\vec{v}(x, y)$ telling how fast the water is flowing and in which direction. $\vec{v}(x, y)$ is a velocity field; it is a vector field. An **electric field** is also a vector field. At each point the electric field $\vec{E}(x, y, z)$ is the force that a positive unit magnitude test charge q_0 feels. A test charge is a spy charge. It feels the force of the other charges but they don't feel it. (Test charges are always positive.) If $q_0 \neq 1$, then we just divide $\vec{F}$ by q_0 to get $\vec{E}$, thus

$$\vec{E}(x, y, z) = \frac{1}{q_0}\vec{F}(x, y, z)$$

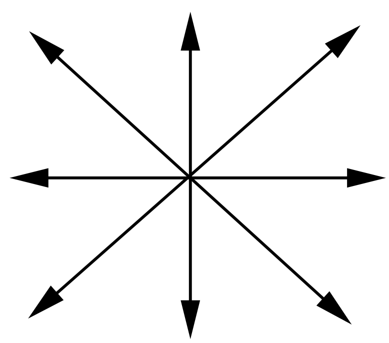
Notice that the value of $\vec{E}$ is independent of the test charge q_0. We can think of $\vec{E} = \frac{\vec{F}}{q_0}$ as the force per unit charge. $\vec{E}$ points in direction that a $\oplus$ wants to go. So it points away from $\oplus$ and toward $\ominus$. $\vec{E}$ is analogous to $\vec{g}$ for gravity: $\vec{E} = \frac{\vec{F}}{q}$ is like $\vec{g} = \frac{\vec{F}}{m}$. When treating an electric field, you should think of the charges as nailed down.

Field of a point charge

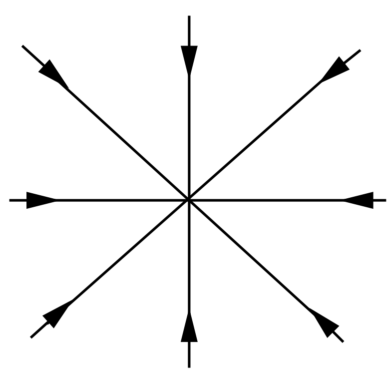
Since Coulomb's law says that the force exerted on q_0 by q is $\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^2}\hat{r}$, the electric field produced by the point charge q is

$$\vec{E} = \frac{\vec{F}}{q_0} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

If q is positive, $\vec{E}$ points radially outward.

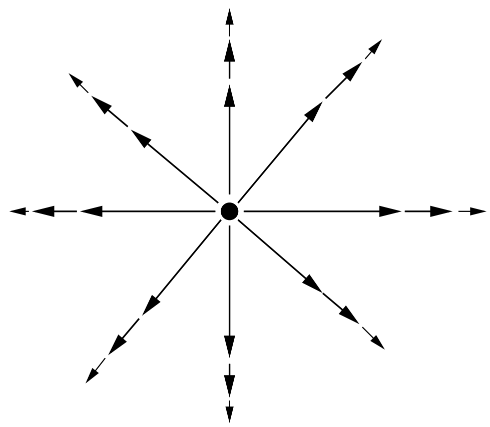


If q is negative, $\vec{E}$ points radially inward.



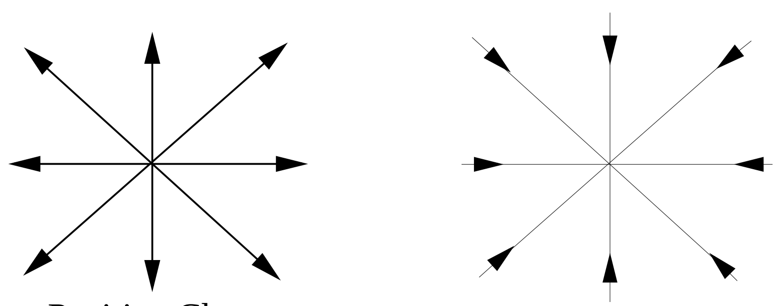
Rules for drawing field lines

How can we visualize the fields. How can we draw them? We could draw vectors to represent what's happening:



But if we try to draw the vectors to scale, we run into problems. The field goes as $1/r^2$ which means that it gets bigger as we get closer to the charge. So if a 1 mm arrow represents the field 10 cm away from the point charge, then we need a 10 cm long arrow to represent the field 1 cm away from the charge.

A somewhat better representation is to connect the arrows to form field lines:



We can't tell the strength from the length of the arrows anymore, but we can from the density of lines. Close in where the field is strong the density is high. In 3 dimensions, we would have a pin cushion with the density of lines decreasing as $1/r^2$. These are some rules you should follow in drawing field line:

1. Decide how many lines for each charge; e.g., 8 lines for $q \Rightarrow$, 16 lines for $2q$.
2. They should emanate symmetrically from a point charge.
3. Positive charges have outgoing lines. Negative charges have incoming lines.
4. Field lines don't stop in midair but they can go out to ∞ or end at a conducting surface.

5. Field lines can't cross. If they could, the field would have 2 vectors representing one point.

Superposition

We've seen the electric field produced by a point charge. What is the field produced by more than one charge? According to the principle of superposition, the force that a test charge q_0 feels is the sum of the forces produced by each of the real changes:

$$\vec{F}_0 = \vec{F}_{0\leftarrow 1} + \vec{F}_{0\leftarrow 2} + \vec{F}_{0\leftarrow 3} + \cdots + \vec{F}_{0\leftarrow n}$$

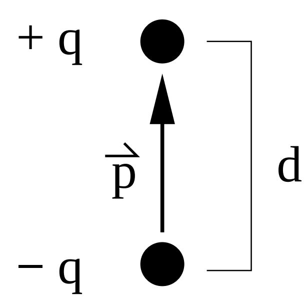
There are n point charges. So the electric field $\vec{E}$ is given by

$$\begin{aligned} \vec{E} &= \frac{\vec{F}_0}{q_0} = \frac{\vec{F}_{0\leftarrow 1}}{q_0} = \frac{\vec{F}_{0\leftarrow 2}}{q_0} + \frac{\vec{F}_{0\leftarrow 3}}{q_0} + \cdots + \frac{\vec{F}_{0\leftarrow n}}{q_0} \\ &= \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \cdots \vec{E}_n \end{aligned}$$

where $\vec{E}_i$ is the electric field that would be set up by point charge q_i acting alone.

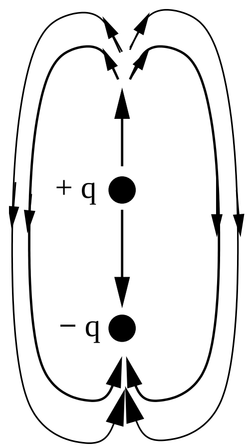
Electric Dipole

As a simple example let's consider 2 point charges: $+q$ and $-q$ $(q > 0)$ separated by a distance d. Both charges lie on the z axis. This charge configuration is called an **electric dipole**. It has an **electric dipole moment** $\vec{p}$ which is a vector whose magnitude is qd. The direction points from the negative charge to the positive charge. So $\vec{p} = qd\hat{z}$ in this case.



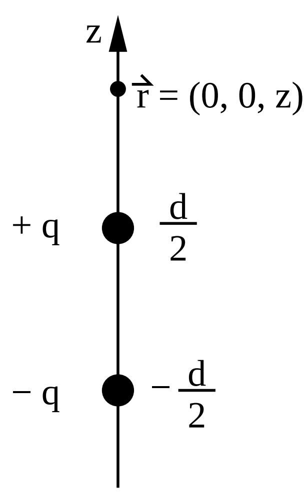
In general the dipole moment's magnitude is the charge times the distance between the charges. (Notice that $\vec{p}$ points opposite to $\vec{E}$.)

What is the electric field at a point along the z-axis? First let's determine the direction of $\vec{E}$. If we put a positive test charge q_0 above the dipole, it is closer to $+q$ than to $-q$. So it feels the repulsion of $+q$ more than the attraction of $-q$. So $\vec{E} \parallel +\hat{z}$. If we place q_0 below the dipole on the z-axis, it feels the attraction of $-q$ more than the repulsion of $+q$ because it's closer to $-q$. So it is attracted to the dipole. The force and hence the electric field is in the $+\hat{z}$ direction. The field of a dipole looks like:



E field of a dipole

Notice that *on* the z-axis $\vec{E} \parallel +\hat{z}$. But away from it the field lines are curved. How do we know that a point on z-axis the field has $\vec{E} \parallel \hat{z}$? Why wouldn't $\vec{E}$ tilt one way or the other? The answer is by symmetry. If it did tilt, which way would it tilt? Right? Left? Backwards? Forwards? Nothing in the problem favors any of these directions. So straight along the z-axis is the direction of $\vec{E}(\vec{r}) = \vec{E}(0,0,z)$. But if our point of observation $\vec{r}$ is to the right $\vec{r} = (x,0,z)$, then $\vec{E}$ could bend to the right $\vec{E}(\vec{r}) = (E_x, 0, E_z)$. In other words our point of observation breaks the symmetry.



Now let's calculate the magnitude of $\vec{E}(\vec{r})$ for a point $\vec{r} = (0,0,z)$ on the z-axis. By superposition we add the fields due to each charge:

$$\begin{aligned} \vec{E} &= \vec{E}_+ + \vec{E}_- \\ &= \frac{1}{4\pi\varepsilon_0}\frac{q_+}{r_+^2}\hat{z} + \frac{1}{4\pi\varepsilon_0}\frac{q_-}{r_-^2}\hat{z} \qquad q_+ = q,\ q_- = -q \\ &= \frac{1}{4\pi\varepsilon_0}\frac{q}{(z-\frac{1}{2}d)^2}\hat{z} + \frac{1}{4\pi\varepsilon_0}\frac{(-q)}{(z+\frac{1}{2}d)^2}\hat{z} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4\pi\varepsilon_0}\frac{q\hat{z}}{z^2(1-\frac{1}{2}\frac{d}{z})^2} - \frac{1}{4\pi\varepsilon_0}\frac{q\hat{z}}{z^2(1+\frac{1}{2}\frac{d}{z})^2} \\
&= \frac{1}{4\pi\varepsilon_0}\frac{q}{z^2}\left[(1-\frac{1}{2}\frac{d}{z})^{-2} - (1+\frac{1}{2}\frac{d}{z})^{-2}\right]\hat{z}
\end{aligned}
$$

Let's assume that the point of observation is far away from the dipole so that $z >> d \Rightarrow \frac{d}{2z} << 1$. Then we can expand the terms in [ ] by the binomial theorem (or equivalently, the Taylor expansion). Recall that the binomial theorem states

$$(x+y)^n = x^n + nx^{n-1}y + \cdots$$

So let $x > 1, y = \pm\frac{1}{2}\frac{d}{z}, n = -2$. Then we get

$$
\begin{aligned}
\vec{E} &= \frac{q}{4\pi\varepsilon_0 z^2}\left[(1-(-2)\frac{d}{2z}+\cdots) - (1+(-2)\frac{d}{2z}+\cdots)\right]\hat{z} \\
&= \frac{q}{4\pi\varepsilon_0 z^2}\left[(\cancel{1}+\frac{d}{z}+\cdots) - (\cancel{1}-\frac{d}{z}+\cdots)\right]\hat{z} \\
&= \frac{q}{4\pi\varepsilon_0 z^2}\frac{2d}{z}\hat{z}\vec{E}(0,0,z) = \frac{qd}{2\pi\varepsilon_0 z^3}\hat{z}
\end{aligned}
$$

If we plug in $\vec{p} = qd\hat{z}$, we get

$$\vec{E}(0,0,z) = \frac{\vec{p}}{2\pi\varepsilon_0 z^3} \quad \text{dipole}$$

Notice that if we had only a point charge $+q$ located at the origin, the electric field along the z-axis would be

$$\vec{E}(0,0,z) = \frac{1}{4\pi\varepsilon_0}\frac{q}{z^2}\hat{z}$$

Notice that the dipole's electric field falls off faster with distance ($E \sim 1/z^3$) than a single point charge ($E \sim 1z^2$). This is because far away from the dipole, the electric field of the minus charge kind of cancels the electric field of the positive charge: $-q+q=0$.

Continuous Charge Distributions

Sometimes we have continuous charge distributions rather than discrete charges. One way to think of a continuous charge distribution is to imagine charged paint, i.e., paint that has lots of positive (or negative) point charges dissolved in it.

If we paint a line or a ring then the amount of charge per unit length

$$\lambda = \frac{dq}{ds}$$

where ds is a tiny segment of the ring or line (i.e., a differential element of length). dq is the charge on ds. λ is the charge density. If we paint a surface (like a wall), then the charge density

$$\sigma = \frac{dq}{dA}$$

If we have a bucket full of charged paint, then the charge density ρ is the amount of charge per unit volume:

$$\rho = \frac{dq}{dV}$$

Recipe to find $\vec{E}$ from continuous charge distribution

Typical Problem: Given $q(\vec{r})$, Find $\vec{E}$.

Recipe for solution:

1. Divide the charge into pieces with charge dq
2. dq produces a field $dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}$
3. Find components of $d\vec{E}$, e.g., $dE_x = |dE|\cos\theta$
4. $\vec{E}_{tot} = \int d\vec{E}$, i.e.

$$\begin{aligned} E_{tot,x} &= \int dE_x \\ E_{tot,y} &= \int dE_y \\ E_{tot,z} &= \int dE_z \end{aligned} \tag{1}$$

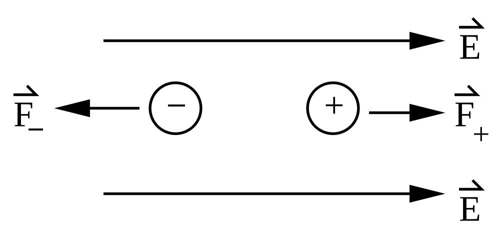
An example of applying this recipe to the problem of finding the electric field produced by a ring of charge is given in the appendix.

Point Charge in a Uniform $\vec{E}$ Field

So far we have been considering the electric field set up by charges or charge distributions. What happens if a charge finds itself in an electric field created by other charges? Answer: the charge feels a force

$$\vec{F} = q\vec{E} \tag{2}$$

Notice that the direction of the force depends on the sign of q. $\vec{E}$ always points in the direction a positive charge wants to go.

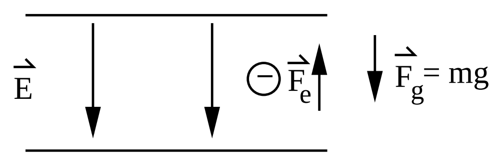


We refer to $\vec{E}$ as the external field since it is not produced by q but rather, acts on q. If the charge is free to move, then it will accelerate according to

$$\vec{a} = \frac{\vec{F}}{m} = \frac{q}{m}\vec{E}$$

Millikan Oil Drop Experiment

Millikan used this to prove that charge was quantified. He shot charged oil drops into an $\vec{E}$ field to counteract the gravitational field. In other words, he used the field to stop the drop from falling.



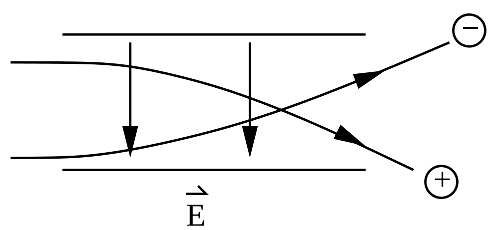
$$F_{tot} = F_g + F_e = mg(-\hat{z}) + q\vec{E} = 0$$

He knew m and g. (He knew the oil's density and could measure its size in a microscope.) So by measuring how big a field he needed, he could deduce the charge. More precisely, he found that charge was quantized.

$$q = ne, \quad \text{where } n = 0, \pm 1, \pm 2, \pm 3 \ldots$$

Ink Jet Printing

A moving particle is affected in the same way by an $\vec{E}$ field: $\vec{F} = q\vec{E}$. Thus a charged particle passing through an $\vec{E}$ field is deflected.



Which particle is positively charged and which is negatively charged?

Gauss' Law

Gauss' law is a way of formulating Coulomb's law that makes it easy to find $\vec{E}$ given a charge distribution $Q(\vec{r})$ if the charge distribution has some symmetry, e.g. spherical, cylindrical, planar, etc. To understand Gauss' Law, we need to understand what flux is.

Flux

Recall that when we talked about field lines, we said that a high density of field lines meant $\vec{E}$ was strong and a low density meant that $\vec{E}$ was weak. The flux through a surface is proportional to the number of field lines piercing a surface. Here are some analogies:

- Think of a pin cushion (with very long pins or spikes) surrounded by a balloon. The number of pins piercing the balloon is the "flux" through the balloon.
- Think of a bed of nails (long nails). If a sheet of plastic is stretched over the bed, the number of nails piercing the sheet is the flux through the sheet. Notice that if the sheet is perpendicular to the nails, you get a lot of flux. If the sheet is parallel to the nails, you get no flux. So the angle of the sheet with respect to the nails matters.
- Suppose it is raining quarters (or nickels) and you get all the quarters that pass through a hula-hoop: Would you hold the loop parallel or perpendicular to the shower? What about tilting the hoop?
- Think of a light bulb that's surrounded by a plastic bag that's perfectly transparent. The amount of light (number of photons) that goes through the bag per second is the flux through the bag. The brighter the light, the more the flux. Also you get the same flux regardless of the shape of the bag.
- A pipe with a screen over the end of it. The amount of water flowing through the screen in 1 second is proportional to the flux through the screen.

For an $\vec{E}$ field, we can think of field lines piercing a surface. We want the component of $\vec{E} \perp$ to the surface. The official definition of flux Φ is

$$\Phi = \int_{\text{surface}} \vec{E} \cdot d\vec{a}$$

Perhaps it's easier to think of a sum over pieces of the surface. Think of dividing the surface into pieces Δa. Each piece is so small that it can be considered flat. $\Delta\vec{a}$ is a vector whose magnitude is the area of the piece and whose direction is perpendicular to the surface. ("Perpendicular" to the surface is also called "normal" to the surface.)

If an electric field passes through this surface, then

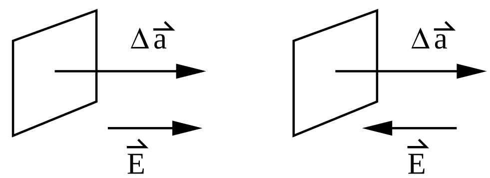
$$\Phi = \sum \vec{E} \cdot \Delta\vec{a} \tag{3}$$

Recall

$$\vec{E} \cdot \Delta\vec{a} = E\Delta a \cos\theta \tag{4}$$

The dot product picks out the component of $\vec{E}$ perpendicular to the surface. If $\vec{E}$ is perpendicular to the surface, then $\vec{E}$ is parallel to $\Delta\vec{a}$ because $\Delta\vec{a}$ is perpendicular to the surface. So $\vec{E} \cdot \Delta\vec{a} = E\Delta a \cos\theta = E\Delta a$ because $\theta = 0$. But if $\vec{E}$ is parallel to the surface, then $\vec{E} \perp \Delta\vec{a} \Rightarrow \vec{E} \cdot \Delta\vec{a} = 0$. So the maximum flux occurs when $\vec{E}$ is perpendicular to the surface, just as in our examples.

Notice that the flux has different signs in the following

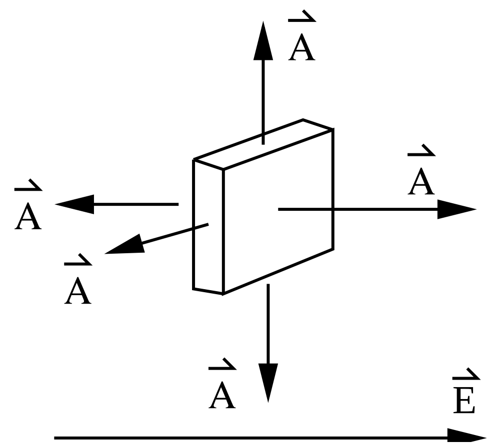


In the limit that the area elements $\Delta\vec{a}$ become infinitesimal elements da, the sum becomes an integral.

$$\Phi = \int_{\text{surface}} \vec{E} \cdot d\vec{a}$$

(The book has capital“$\vec{A}$” : $\Phi = \int_{\text{surface}} \vec{E} \cdot d\vec{A}$.)

Calculate the net flux through a thin box. $d\vec{A}$ points outward.



$$\Phi = \underbrace{EA}_{front} + \underbrace{(-EA)}_{back} = 0$$

No contribution from sides (5)

what comes in one side goes out the other $\Longrightarrow$ no net flux.

Gauss' Law deals with the flux ϕ through closed surfaces:

$$\Phi = \oint \vec{E} \cdot d\vec{A} \tag{6}$$

where $\oint$ means that we should integrate over a closed surface, e.g., a box or a closed bag or a balloon, etc.

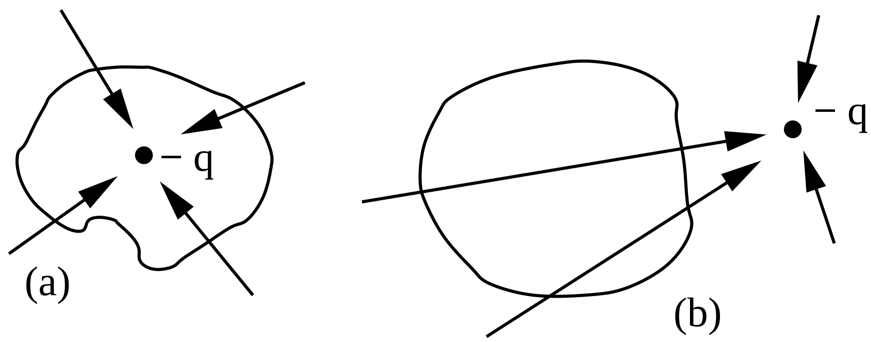
Gauss' Law says

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{enc} \tag{7}$$

where q_{enc} is the total amount of charge enclosed by the surface.

Suppose an imaginary or "Gaussian" surface encloses some blob of charge q_{enc}. Gauss' law says that the total flux through the surface is proportional to the charge enclosed. q_{enc} is the total amount of charge enclosed. Notice that $\Phi < 0$ if $q < 0$ and $\Phi > 0$ if $q > 0$.

Examples:



$$\varepsilon_0 \oint_{S1} \vec{E} \cdot d\vec{A} = -q \qquad\qquad \varepsilon_0 \oint_{S2} \vec{E} \cdot d\vec{A} = 0$$

Notice that the more charge that is enclosed, the more flux there is. This is like the light bulb in the bag – brighter light means more flux.

In (b), notice that the flux $\Phi = 0$ even though $\vec{E} \neq 0$.

Gauss' Law is a useful trick for finding $\vec{E}$ if you are given a symmetrical charge distribution. If the distribution is unsymmetrical, it's too hard to do the integral $\oint \vec{E} \cdot d\vec{A}$. But for certain symmetrical distributions, you can choose a Gaussian surface so that you don't really have to do an integral. In some cases, the integral winds up being zero because the field $E = 0$ or because $\vec{E} \perp d\vec{a}$. In other cases the E is constant on the Gaussian surface and the integral $\oint \vec{E} \cdot d\vec{a} = EA$. The symmetries where this happens are spherical, cylindrical, and planar. What follows are the easy examples of using Gauss' law to find the $\vec{E}$ field.

First let's go over the basic strategy for solving problems using Gauss' Law.

Recipe for Solving Problems with Gauss' Law

Typical Problem: Given the charge distribution $q(\vec{r})$, find $\vec{E}(\vec{r})$.

Recipe:

1. See if charge distribution has some symmetry, e.g., cylindrical, spherical, planar. If so, use Gauss' Law. If not, use Coulomb's Law and principle of superposition.

2. Determine the direction of $\vec{E}$.

3. Draw a closed Gaussian surface that matches the symmetry of the charge. Try to make the surface such that $\vec{E} \parallel d\vec{a}$ or $\vec{E} \perp d\vec{a}$ on the different sides. Make sure the surface encloses the charge that produces the field you want to calculate. You want $\vec{E}$ parallel to a single coordinate like $\hat{x}$ or $\hat{r}$.

4. Evaluate $\oint \vec{E} \cdot d\vec{a}$.

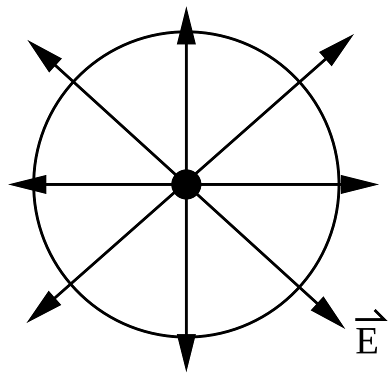
5. Calculate q_{enc}.

6. Solve $\varepsilon_0 \oint \vec{E} \cdot d\vec{a} = q_{enc}$ for $\vec{E}$.

Let's apply this recipe to some examples.

Point Charge

As an example, suppose we are given a point charge $q > 0$. What is $\vec{E}$? The point charge and its field $\vec{E}$ are spherically symmetric. So let's surround it with a spherical Gaussian surface of radius r.



Gauss' Law says

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = q \tag{8}$$

The point charge is in the center. The field $\vec{E}$ points radially outward $\Rightarrow \vec{E} \cdot d\vec{A} = E dA$.

$$\varepsilon_0 \oint E \, dA = q \tag{9}$$

E is a function of the radial distance r. It is therefore a constant on the sphere's surface. So pull E out of the integral.

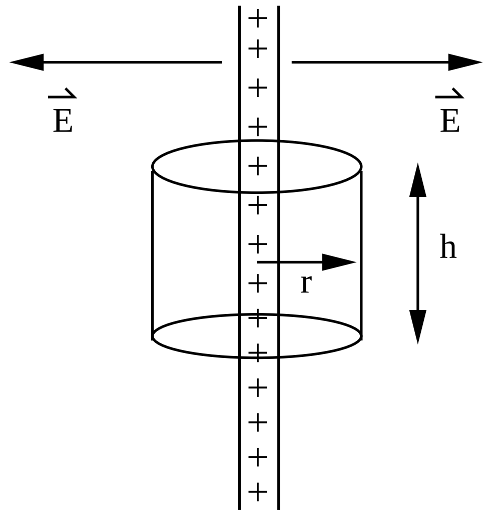
$$\varepsilon_0 E \oint dA = q \quad \Longrightarrow \quad \varepsilon_0 E A = q$$

$$A = 4\pi r^2 \quad \Longrightarrow \quad \varepsilon_0 E (4\pi r^2) = q \Longrightarrow E = \frac{q}{\varepsilon_0 4\pi r^2} \tag{10}$$

This is exactly what we got from Coulomb's Law. In fact Gauss' law is equivalent to Coulomb's Law.

Line of Charge

Consider a straight, infinitely long line of positive charge with a charge per unit length λ. Find $\vec{E}$ a distance r from the line.

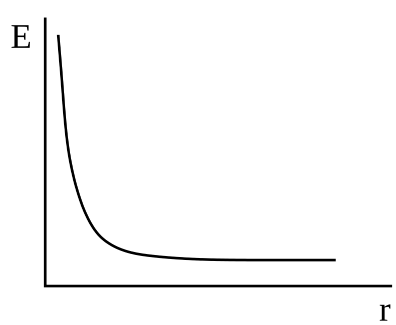


Solution: Note the cylindrical symmetry and $\vec{E} \parallel \hat{r}$ ($\vec{E}$ points radially outward). The Gaussian surface is a can of radius r and height h. $\vec{E} \parallel d\vec{a}$ on side. $\vec{E} \perp d\vec{a}$ on the top and the bottom. $\vec{E}$ at a distance r is a constant.

$$\begin{aligned}
\oint \vec{E} \cdot d\vec{a} &= \int_{\text{side}} \vec{E} \cdot d\vec{a} = E \int_{\text{side}} da = E(2\pi r h) \\
q_{enc} &= \lambda h \\
\varepsilon_0 \oint \vec{E} \cdot d\vec{a} &= q_{enc} \Longrightarrow \varepsilon_0 E(2\pi r \not{h}) = \lambda \not{h} \\
E &= \frac{1}{\varepsilon_0} \frac{\lambda}{2\pi r} \qquad \text{which decreases as } \frac{1}{r} \\
\text{or} & \\
\vec{E} &= \frac{1}{\varepsilon_0} \frac{\lambda}{2\pi r} \hat{r} \qquad \vec{E} \text{ points radially outward}
\end{aligned}$$

If $\lambda < 0$,

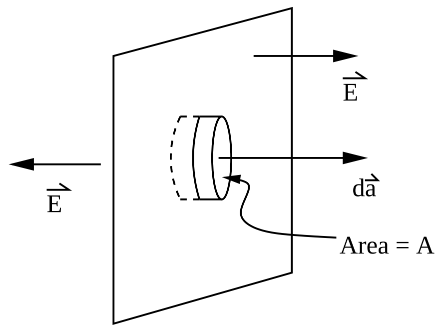
$$\vec{E} = \frac{1}{\varepsilon_0} \frac{\lambda}{2\pi r} (-\hat{r})$$



Plane of Charge

Consider an infinite plane of positive charge with uniform surface charge density σ. Assume the sheet is insulating so that the charge stays fixed. Find $\vec{E}$ a distance r from the sheet.

Solution: Planar symmetry. $\vec{E}$ points away from the sheet. $\vec{E} \perp$ to the sheet. Draw Gaussian pillbox with ends parallel to the sheet such that $\vec{E} \parallel d\vec{a}$ at ends. No flux through sides.



Note that E a distance r from the sheet is constant, i.e. it is constant on the end face of the box. The area of the end of the box is A, so the flux through that face is EA. There are 2 faces, so the total flux is $\Phi = 2EA$. According to Gauss' law,

$$\Phi = 2EA = \frac{q_{enc}}{\varepsilon_0} \tag{11}$$

The charge enclosed is

$$q_{enc} = \sigma A \tag{12}$$

So we have

$$\Phi = 2EA = \frac{\sigma A}{\varepsilon_0} \tag{13}$$

or

$$E = \frac{\sigma}{2\varepsilon_0} \tag{14}$$

Note: No dependence on distance from sheet.

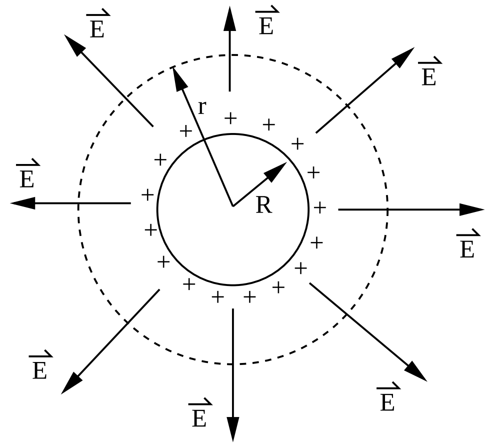
Spherical Shell of Charge:

Consider a spherical shell of uniform charge density. Let $q > 0$ be the total charge of the shell. The shell has radius R. Find $\vec{E}(r)$ for $r > R$ (outside) and $r < R$ (inside).

Solution: Spherical Symmetry

Outside ($r > R$):

$\vec{E}$ points radially outward in $\hat{r}$ direction. Gaussian surface is concentric spherical surface outside the shell of charge. Gaussian sphere has radius r. E is constant on the sphere so we can take it out of the integral.

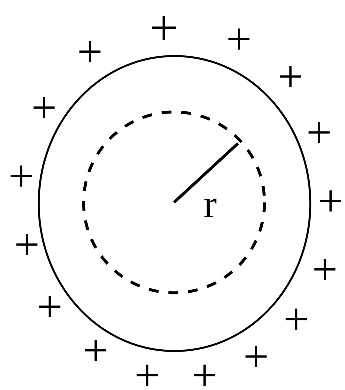


$$\oint \vec{E} \cdot d\vec{a} = E \oint da = E \cdot 4\pi r^2$$
$$q_{enc} = q$$
$$\varepsilon_0 \oint \vec{E} \cdot d\vec{a} = q_{enc} \Rightarrow \varepsilon_0 E \cdot 4\pi r^2 = q$$
$$E = \frac{q}{4\pi\varepsilon_0 r^2} \qquad r > R$$

This is the same field as a point charge q at the center of the sphere. Thus "a shell of uniform charge attracts or repels a charged particle that is outside the shell as if all the shell's charge were concentrated at the center of the shell."

Inside ($r < R$):

By symmetry, if $\vec{E}$ points in any direction, it will be radial, i.e., along with $\hat{r}$. Draw a Gaussian sphere inside the shell. Gaussian sphere has a radius r. E is a constant on the Gaussian sphere so we can take it out of the integral.



$$\oint \vec{E} \cdot d\vec{a} = E \cdot 4\pi r^2 \tag{15}$$

$$q_{enc} = 0 \tag{16}$$

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{a} = q_{enc} \Longrightarrow \varepsilon_0 E \cdot 4\pi r^2 = 0 \Longrightarrow E = 0 \quad r < R \tag{17}$$

So $\vec{E} = 0$ inside a spherical shell of uniform charge. So if we have a charged particle inside the shell, it feels no electrostatic force due to the shell, because $\vec{E} = 0$ inside the shell. $\vec{E} = 0$ because when we add up the contributions to $\vec{E}$ from different parts of the shell, they all cancel out. This is easiest to see in the center but it's true everywhere inside the shell.

Solid Sphere of Charge

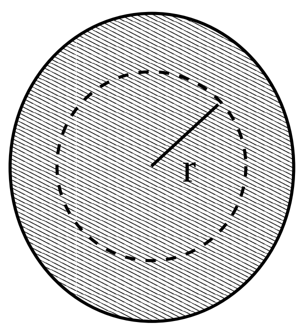
Suppose we have a solid sphere of uniform charge density. The radius of the sphere is R. The total charge contained in the sphere is $Q > 0$. Find $\vec{E}$ both outside $(r > R)$ and inside $(r < R)$ the sphere.

One Solution: Spherical symmetry. Divide the sphere into spherical shells and use superposition to add up the contributions to $\vec{E}$ from the shells.

Outside $(r > R)$: $\vec{E}$ is the same as for a point charge Q at the center of the sphere.

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

Inside $(r < R)$:

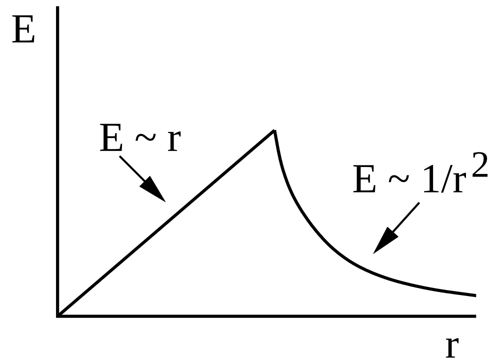


Consider a spherical Gaussian surface of radius r inside the charged ball. $E(r)$ only has contributions from charge inside the Gaussian surface. Call this charge q'. The field at r is the same as for a point charge q' at the center. So

$$\vec{E}(r) = \frac{q'}{4\pi\varepsilon_0 r^2}\hat{r}$$

What is q' in terms of Q, the total charge? Q and q' are proportional to the volume since the charge density is uniform, so

$$\begin{aligned} \rho = \frac{q'}{\frac{4}{3}\pi r^3} &= \frac{Q}{\frac{4}{3}\pi R^3} \Rightarrow q' = Q\frac{r^3}{R^3} \\ E(r) &= \frac{Q}{4\pi\varepsilon_0 \cancel{r^2}}\frac{r^{\cancel{3}}}{R^3} = \frac{Q}{4\pi\varepsilon_0 R^3}r \quad r < R \end{aligned}$$



Another Solution Calculate q_{enc} using the uniform charge density

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3} \Rightarrow q_{enc} = \int dV\,\rho$$

Outside $(r > R)$ Draw a spherical Gaussian surface. $\vec{E}$ is radial and points in the $\hat{r}$ direction. So

$$\begin{aligned}
\oint \vec{E}\cdot d\vec{a} &= E(4\pi r^2) \\
q_{enc} &= \int \rho dV = \rho \int dV = \rho\cdot\frac{4}{3}\pi R^3 = \left(\frac{Q}{\frac{4}{3}\pi R^3}\right)\left(\frac{4}{3}\pi R^3\right) \\
&= Q \\
\varepsilon_0 \oint \vec{E}\cdot d\vec{a} &= q_{enc} \quad\Rightarrow\quad \varepsilon_0 E\cdot 4\pi R^2 = Q \\
E &= \frac{Q}{4\pi\varepsilon_0 r^2} \quad (r > R) \quad \text{as before}
\end{aligned}$$

Inside $(r < R)$: Spherical Gaussian surface inside the ball of charge.

$$\begin{aligned}
\text{flux} &= \oint \vec{E}\cdot d\vec{a} = E(4\pi r^2) \\
q_{enc} &= \int \rho dV = \rho\cdot\frac{4}{3}\pi r^3 = \frac{Q}{\frac{4}{3}\pi R^3}\cdot\frac{4}{3}\pi r^3 \\
&= Q\frac{r^3}{R^3} \\
\varepsilon_0 \oint \vec{E}\cdot d\vec{a} &= q_{enc} \Longrightarrow \varepsilon_0 E\cdot(4\pi r^2) = Q\frac{r^3}{R^3} \\
E &= \frac{Q}{4\pi\varepsilon_0 r^2}\frac{r^3}{R^3} \\
E &= \frac{Q}{4\pi\varepsilon_0 R^3} r \quad r < R
\end{aligned}$$

This is the same answer we got before.

Charged Isolated Conductor

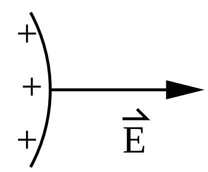
An isolated conductor can hold charge because there's nowhere for the charge to go. If we put excess charge on a conductor, it resides on the surface of the conductor, not in the interior. How do we know this? Because $\vec{E} = 0$ **everywhere inside a conductor**. If $\vec{E} \neq 0$ somewhere inside, then the free charges (conduction electrons) would feel a force $\vec{F} = q\vec{E}$ and they would move in response to the force. But an isolated conductor doesn't have flowing charges. So $\vec{E} = 0$ inside. Gauss' Law tells us that

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{a} = q_{enc} = 0$$

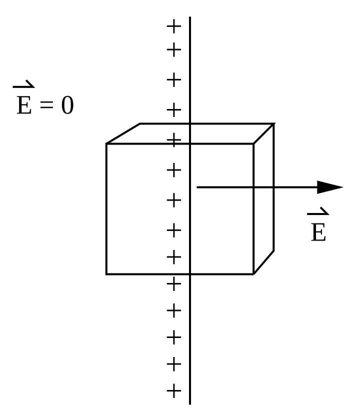
If the Gaussian surface lies inside the conductor, $\vec{E} = 0$ on the gaussian surface which implies no excess charge is enclosed. (There can be charge enclosed, but it must consist of equal amounts of positive and negative charge.) So all the excess charge is on the surface.

When you first dump charge on a conductor, it runs around until all the forces balance out. The charges, which all have the same sign, try to get as far apart as possible. When they get to the surface, they've gone as far as they can go. So they stop.

Notice that $\vec{E}$ is perpendicular to the surface. If $\vec{E}$ had any components tangent to the conductor's surface, the charge would run along the surface.



Let's suppose that a charged isolated conductor has a surface charge density of $\sigma(\vec{r})$. For an irregularly shaped conductor, $\sigma(\vec{r})$ may vary along the surface. Let's find $\vec{E}$ at the surface. Consider a small element of surface - small enough to be flat and to have $\sigma(\vec{r})$ = const. Draw a Gaussian pillbox. $\vec{E} \perp$ surface. $E = 0$ inside the conductor, so we only get flux through the "front" side of the box. (The "front" is parallel to the conductor's surface.)

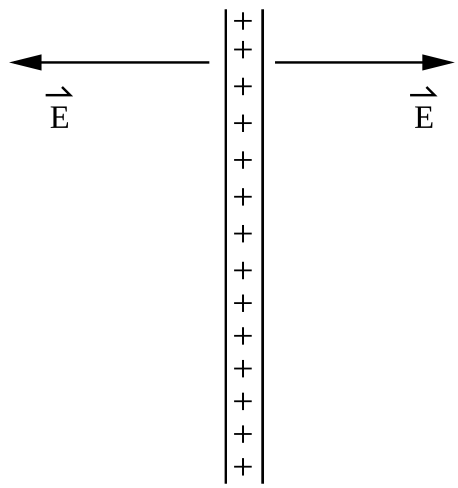


$$\text{Flux} = \oint \vec{E} \cdot d\vec{a} = EA \tag{18}$$

where A is the area of the front of the box. Here we assume E is a constant on the front of the box because the Gaussian pillbox is very small.

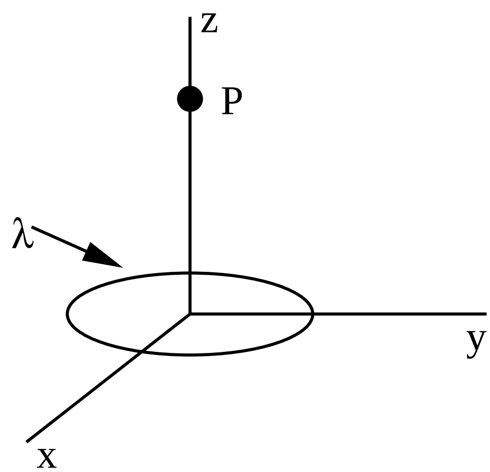
$$\begin{aligned} q_{enc} &= \sigma A \\ \varepsilon_0 \oint \vec{E} \cdot d\vec{a} &= q_{enc} \Rightarrow \varepsilon_0 E \not{A} = \sigma \not{A} \rightarrow E = \frac{\sigma}{\varepsilon_0}. \end{aligned} \tag{19}$$

Notice that this is different from the charged insulating sheet which had $\vec{E}$ going out the front and back, giving $E = \frac{\sigma}{2\varepsilon_0}$.



Appendix

Charged Ring Problem:
Find the electric field a distance z above the center of a circular ring of radius P which carries a uniform charge density of λ.



Solution:

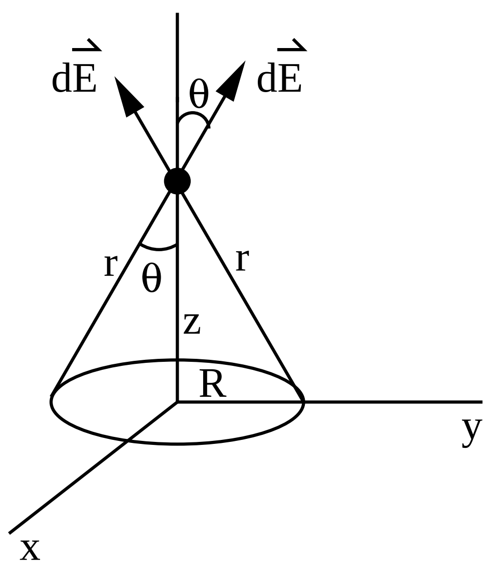
First let's ask, "what do we expect by symmetry?" There is no preferred x or y direction. The system has azimuthal symmetry, i.e., if we rotate the ring in the x-y plane about the z-axis, things are the same. So $\vec{E}(P)$ has no x or y component: $E_x = E_y = 0$. If $\vec{E}$ did have a component in the x-y plane, which way would it point without showing favoritism? Another way to see this is to note that the charge on opposite sides of the circle produce fields whose x and y components cancel. So $\vec{E}$ is parallel to the $+\hat{z}$.

Calculate $\vec{E}(P) = E_z\hat{z}$. To do this, we use the principle of superposition. We divide the ring into segments, each of length ds and charge $dq = \lambda ds$. Then we calculate the field $d\vec{E}$ due to this segment. Finally we add up all the fields to get the total field produced by the ring.

$$\vec{E} = \oint d\vec{E}$$

So the magnitude of dE is given by Coulomb's law

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}$$



Plug in $dq = \lambda ds$ and $r^2 = R^2 + z^2$ to get

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{\lambda ds}{R^2 + z^2}$$

Since the total $\vec{E} \parallel \hat{z}$, we just want the z-component of $d\vec{E}$:

$$dE_z = dE\cos\theta$$

We can express $\cos\theta$ in terms of R and z:

$$\cos\theta = \frac{z}{r} = \frac{z}{[R^2+z^2]^{\frac{1}{2}}} \tag{20}$$

So

$$\begin{aligned} dE_z &= dE\cos\theta \\ &= \frac{1}{4\pi\varepsilon_0}\frac{\lambda ds}{R^2+z^2}\cdot\frac{z}{[R^2+z^2]^{\frac{1}{2}}} \\ &= \frac{z\lambda}{4\pi\varepsilon_0[z^2+R^2]^{\frac{3}{2}}}\,ds \end{aligned}$$

To get the total field, we integrate over dE_z:

$$E_z = \int dE_z = \frac{z\lambda}{4\pi\varepsilon_0[z^2+R^2]^{\frac{3}{2}}}\oint ds$$

Only ds varies as we go around the ring, so only ds stays inside the integral.

$$\oint ds = 2\pi R \tag{21}$$

which is the circumference of the ring. So

$$E_z = \frac{z\lambda(2\pi R)}{4\pi\varepsilon_0[z^2+R^2]^{\frac{3}{2}}} \tag{22}$$

Notice that $q = \lambda(2\pi R)$ is the total charge on the ring.

$$E_z = \frac{qz}{4\pi\varepsilon_0(z^2+R^2)^{\frac{3}{2}}} \tag{23}$$

or

$$\vec{E}(0,0,z) = E_z\hat{z} = \frac{qz}{4\pi\varepsilon_0(z^2+R^2)^{\frac{3}{2}}}\,\hat{z} \tag{24}$$

Notice that far from the ring ($z \gg R$), $z^2 + R^2 \approx z^2$ and

$$\vec{E}(0,0,z) \cong \frac{qz}{4\pi\varepsilon_0 z^3}\hat{z} = \frac{q}{4\pi\varepsilon_0 z^2}\hat{z} \qquad (z >> R)$$

This is the field of a point charge. So far from the ring, the ring looks like a point charge. (It's a good idea to take limits to see if we get sensible results).

If $z = 0$, $\vec{E}(0,0,z=0) = 0$. This says that in the center of the ring, $E = 0$. This is because the field produced by one bit of the ring is cancelled by a bit of charge on the opposite side of the ring.