Lecture 1

Motivation for course

The title of this course is "condensed matter physics" which includes solids and liquids (and occasionally gases). There are also intermediate forms of matter, e.g., glasses, rubber, polymers, and some biophysical systems. Basically this is the branch of physics that covers the things we see and touch in everyday life, i.e., "real stuff." Most of the materials we meet in every day life are amorphous, but since we understand crystalline materials so much better, that is what we will spend most of our time talking about.

Why should we study condensed matter physics?

1. "Because it's there."

2. Real-life physics

3. Frontier of complexity – "more is different"
   Think of a spin - a multitude gives all sorts of magnetism due to interactions

4. Analogies with elementary particle physics, e.g., Higgs mechanism, topological winding numbers, broken symmetry, etc.

5. Practical applications, e.g., transistors.

Drude Theory of Metals

(a) **Phenomenology of metals** – high electrical conductivity, shiny (reflecting), ductile + malleable, high thermal conductivity, etc. Found generally in columns 1A and 2A of the periodic table, among heavier III-VI column elements, and in transition metals and rare earths. In general, they have 1-2 extra electrons above a closed shell. Typically $\rho_{metal} \sim$ few $\mu\Omega$-cm versus $\rho_{insulator} \sim 10^{17}$ Ω-cm for insulators like polystyrene.

(b) **Basic concepts** - The extra electrons are called conduction electrons and they are free to move within the volume. Core electrons stay home. The number of conduction electrons

$$n_e \cong Z n_{Avagadro} \sim 10^{22} - 10^{23} \,\text{electrons/cm}^3$$

where Z is the chemical valence (see table 1.1 of AM). Electronic density is often defined in terms of r_s = radius of sphere whose volume is equal to the volume per conduction electron:

$$\frac{V}{N} = \frac{1}{n} = \frac{4\pi r_s^3}{3}; \quad r_s = \left(\frac{3}{4\pi n}\right)^{1/3}$$

Typically $r_s \sim 1 - 3\mathring{A}$. Natural unit is Bohr radius $a_0 = \hbar^2/me^2 = 0.529 \times 10^{-8}$ cm.

$$\frac{r_s}{a_0} \sim 2 - 6$$

For comparison, note that a typical atomic (ionic) radius is $\sim 0.3 - 2\mathring{A}$. So conduction electrons occupy a larger sphere than ions.

(c) **Electrical Conductivity** (resistivity)

$$\sigma = \frac{1}{\rho} \qquad \vec{\jmath} = \sigma\vec{E} \qquad \vec{E} = \rho\vec{\jmath}$$

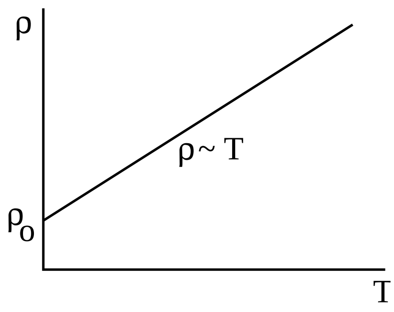
Let A = cross sectional area of wire, L = length,

$$j = \frac{I}{A} \text{ and } V = EL \quad \Rightarrow \quad V = \rho jL = \rho\frac{I}{A}L = IR$$
$$\Rightarrow \quad R = \rho\frac{L}{A} \text{ or } \rho = R\frac{A}{L}$$

Longer wires have more resistance. Larger A means more manuverability for electrons and less resistance. As we said before, $\rho(300°K) \sim 1\mu\Omega$-cm.

At not too low T, $\rho(T) \sim T$ (phonon scattering). As $T \to 0$, $\rho(T) \to \rho_0$ = residual resistivity due to scattering of impurities. This yields Matthiessen's Rule:

$$\rho(T) = \rho_0 + \rho' T \tag{1.1}$$



Assumptions of the Drude model:

(i) Electrons move independently under the influence of local electric field between collisions.

(ii) Collisions are instantaneous, with some unspecified but energy-nonconserving mechanism.

(iii) Collisions are random, with probability dt/τ per unit time (no history dependence).

(iv) Electrons totally thermalized to local temperature by inelastic collisions.

DC Conductivity $(\vec{B} = 0,\ \vec{E} = const)$

$$\text{Electric current } \vec{\jmath} = -\,ne\langle\vec{v}\rangle \tag{1.2}$$

The minus sign is due to the negative charge of the electrons. There are two contributions to $\left.\frac{d\vec{\jmath}}{dt}\right|_{total}$:

$$\left.\frac{d\vec{\jmath}}{dt}\right|_{total} = \left.\frac{d\vec{\jmath}}{dt}\right|_{collision} + \left.\frac{d\vec{\jmath}}{dt}\right|_{field}$$

Field:

$$\begin{aligned} \text{Force} \quad &= \quad \vec{F} = \frac{d\vec{p}}{dt} \Rightarrow \qquad m\frac{d\langle\vec{v}\rangle}{dt} = -e\vec{E} \\ &\Rightarrow \left.\frac{d\vec{\jmath}}{dt}\right|_{field} = -\frac{d(ne\langle\vec{v}\rangle)}{dt} = \frac{ne^2\vec{E}}{m} \end{aligned}$$

Collisions: Collisions knock electrons out of the current flow. So we expect $\left.\frac{d\vec{\jmath}}{dt}\right|_{coll} < 0$: degrade current

$$\begin{aligned} \delta\langle\vec{v}\rangle &= -\langle\vec{v}\rangle \times (\text{prob of collision} \sim \text{fraction of particles affected}) \\ &= -\langle\vec{v}\rangle\frac{dt}{\tau} \quad (\tau \text{ is relaxation time}) \\ \left.\frac{d\langle\vec{v}\rangle}{dt}\right|_{coll} &= -\frac{\langle\vec{v}\rangle}{\tau} \\ \Rightarrow \left.\frac{d\vec{\jmath}}{dt}\right|_{coll} &= -\frac{\vec{\jmath}}{\tau} \end{aligned}$$

So

$$\left.\frac{d\vec{\jmath}}{dt}\right|_{total} = \frac{ne^2\vec{E}}{m} - \frac{\vec{\jmath}}{\tau}$$

In a steady state with $\vec{E} = const$, $\vec{\jmath}$ must be constant:

$$\begin{aligned} \left.\frac{d\vec{\jmath}}{dt}\right|_{total} &= 0 \\ \Rightarrow \qquad \vec{\jmath} &= \frac{ne^2\tau}{m}\vec{E} \\ &= \sigma_0\,\vec{E} \end{aligned}$$

where the DC (E = constant) conductivity is given by

$$\boxed{\sigma_0 = \frac{ne^2\tau}{m}} \qquad \text{sign of charge doesn't matter}$$

(When $\vec{E} = \vec{E}(t)$, $\sigma = \sigma(\omega)$, i.e., the conductivity has frequency dependence.)

From experimental values of σ_0 and n, we can work out τ (see AM, table 1.2). Typically, $\tau \sim 10^{-14} - 10^{-15}$ sec. at room temperature ($\tau^{-1} \sim T$). At low T, $\tau \lesssim 10^{-9}$ sec and is limited by impurity scattering. Matthiessen's rule: $\tau^{-1} \sim \tau_0^{-1} + (\tau^{-1})'T$.

We can define a mean free path $\ell \sim \bar{v}\tau$. How do we estimate $\bar{v}$? Drude used kinetic theory of gases and said

$$\begin{aligned} & \frac{1}{2}m\bar{v}^2 = \frac{3}{2}kT \\ \Rightarrow \quad & \bar{v}_{rms} \sim 10^7\,\frac{cm}{s} \quad \text{at} \quad T = 300K \\ \Rightarrow \quad & \ell \sim 1 - 10\mathring{A} \sim \text{lattice spacing or distance between ions} \end{aligned}$$

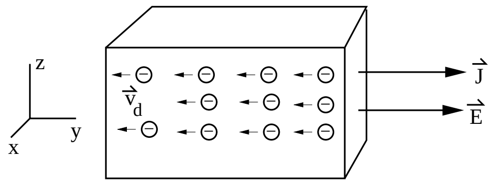
But this is misleading. (Should use $v_F \sim 10^8$ cm/s)

Conduction in a Magnetic Field

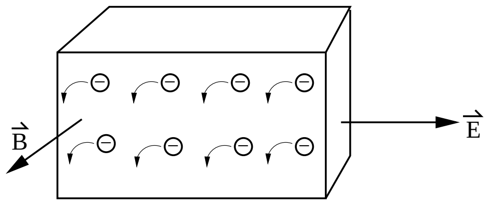
In the presence of a magnetic field $\vec{B}$, an additional Lorentz force acts on the electrons.

$$\vec{F} = -e(\vec{E} + \frac{\vec{v}}{c} \times \vec{B})$$

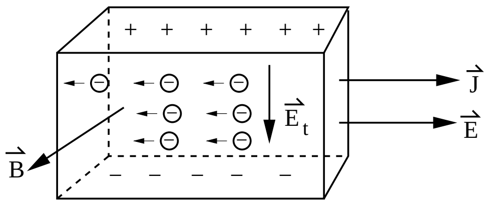
This leads to the *Hall Effect*. Consider a metal bar with current flowing in it carried by electrons with average velocity $\bar{v}$.



Now suppose we apply a magnetic field in the $\hat{x}$ direction. This initially causes a downward deflection of the moving electrons.



Negative charge builds up at the bottom; positive charge at the top. The transverse electric field $\vec{E_t}$ counters the magnetic force so that the electrons again flow in the $-\hat{y}$ direction.



Notice that if the charge carriers had been positively charged, $\vec{E_t}$ would point in the opposite direction ($\vec{\jmath}$ is in the same direction as before). Thus if we measure the voltage difference between top and bottom, the sign should tell us the sign of the carriers. (We expect negative, but sometimes it's positive. More on this later.) It is easy to determine the magnitude of $\vec{E_t}$ by balancing the electric force with the magnetic force in the z-direction. (Let's use q rather than $(-e)$.)

$$q\,E_t = q\frac{\bar{v}}{c}B \Rightarrow E_t = \frac{\bar{v}}{c}B$$

We know $\vec{\jmath} = nq\vec{v} \Rightarrow \bar{v} = j/nq$

$$E_t = \left(\frac{1}{nqc}\right) jB = R_H jB$$

where $\boxed{R_H = \frac{1}{nqc}}$ is called the Hall coefficient.
For $q=-e$, $\quad R_H = -\frac{1}{nec}$
Experimentally, $\boxed{R_H = \frac{E_t}{jB}}$ $\vec{E}_t \perp \vec{\jmath} \perp \vec{B}$.

Note that because $\vec{E}_t$ cancels the effect of the magnetic field, we still have $j_y = \sigma_0 E_y$ (different coords than AM). You can check this by looking at $\left.\frac{d\vec{\jmath}}{dt}\right|_{total}$. Experimentally, this isn't always true. Drude model is too simple.

AC Conductivity

$(\vec{B}=0, \vec{E}(t)) \quad \sigma(\omega)$

Consider an electric field that is varying in time:

$$E(t) = E_0 \cos \omega t = Re\left(E_0\, e^{-i\omega t}\right)$$

The response of the electrons as well as the current will also vary in time. This leads to a frequency dependent conductivity.

$$\vec{\jmath}(t) = Re\left(\vec{\jmath}_0 e^{-i\omega t}\right) \text{ where } \vec{\jmath}_0 = \sigma(\omega)\vec{E}_0$$

In general $\sigma(\omega)$ will be complex, indicating that $\vec{\jmath}$ is out of phase with $\vec{E}$.

Calculate $\sigma(\omega)$

Start with

$$\left.\frac{d\vec{\jmath}}{dt}\right|_{total} = \frac{ne^2\vec{E}(t)}{m} - \frac{\vec{\jmath}}{\tau}$$

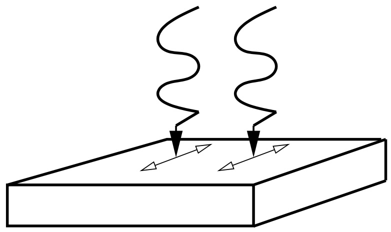
Plug in $\vec{\jmath}(t) = j_0 e^{-i\omega t}$ and $\vec{E}(t) = \vec{E}_0 e^{-i\omega t}$ to get

$$\begin{aligned} -i\omega\vec{\jmath}_0 &= \frac{ne^2\vec{E}_0}{m} - \frac{\vec{\jmath}_0}{\tau} \\ (-i\omega + \frac{1}{\tau})\vec{j}_0 &= \frac{ne^2\vec{E}_0}{m} \\ \vec{j}_0 &= \frac{ne^2}{m}\left(\frac{1}{-i\omega+\frac{1}{\tau}}\right)\vec{E}_0 \end{aligned}$$

$$\boxed{\sigma(\omega) = \frac{ne^2\tau}{m}\left(\frac{1}{1-i\omega\tau}\right) = \frac{\sigma_0}{1-i\omega\tau}}$$

$$\text{If } \vec{E} = \vec{E}_0 \cos\omega t, \text{ then } \vec{\jmath} = \frac{\sigma_0 E_0}{\sqrt{1+\omega^2\tau^2}} \cos(\omega t - \delta)$$

where $\tan\delta = \omega\tau$.



We can relate $\sigma(\omega)$ to the frequency dependent dielectric constant $\varepsilon(\omega)$. Consider a piece of metal that is free-standing. Suppose we irradiate it with electromagnetic radiation. There will be no free current $\vec{\jmath}_f$ but there will be a polarization current because the electrons slosh back and forth:

$$\vec{\jmath} = \frac{\partial \vec{P}}{\partial t} \quad \Rightarrow \quad \vec{P} = \frac{\vec{\jmath}}{-i\omega} = \frac{\sigma\vec{E}}{-i\omega} = \frac{i\sigma\vec{E}}{\omega}$$

$$\vec{D} = \varepsilon\vec{E} = \vec{E} + 4\pi\vec{P} \quad \Rightarrow \quad \varepsilon = 1 + 4\pi\frac{\vec{P}}{\vec{E}} = 1 + \frac{4\pi i\sigma}{\omega}$$

$$\boxed{\varepsilon(\omega) = 1 + \frac{4\pi i\sigma(\omega)}{\omega}}$$

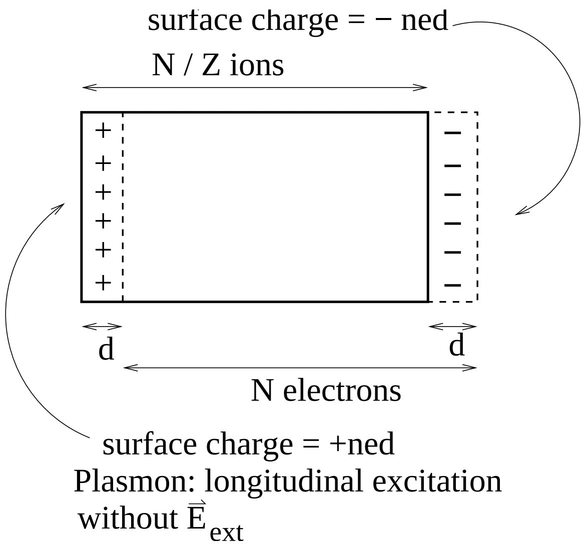
Plasma Frequency $(\omega\tau \gg 1)$

At high frequencies $(\omega\tau \gg 1)$

$$\begin{aligned} \sigma(\omega) &= \frac{\sigma_0}{1 - i\omega\tau} \simeq \frac{i\sigma_0}{\omega\tau} \\ \Rightarrow \varepsilon(\omega) &= 1 - \frac{4\pi\sigma_0}{\omega^2\tau} = 1 - \frac{4\pi}{\omega^2 \cancel{\tau}}\frac{ne^2 \cancel{\tau}}{m} = 1 - \frac{\omega_p^2}{\omega^2} \end{aligned}$$

where $\boxed{\omega_p^2 = \frac{4\pi ne^2}{m}}$. This is called the **plasma frequency**.

What does this mean physically? ω_p is the characteristic frequency for the electrons to slosh back and forth. These are called plasma oscillations, or **plasmons**. AM give a simple model of this. Imagine displacing the entire electron gas, as a whole, through a distance d with respect to the fixed positive background of ions.



The resulting surface charge gives rise to an electric field of magnitude $4\pi\sigma$, where σ is the charge per unit area (recall Gauss' Law). The electron gas obeys the equation of motion

$$\vec{F} = -Ne\vec{E} \quad (\vec{E}_{ext} = 0.\ \text{This } \vec{E} \text{ is internally generated.})$$

$$Nm\ddot{d} = -Ne\,E = -Ne|4\pi\sigma| = -Ne(4\pi nde)$$

$$Nm\ddot{d} = -4\pi ne^2 Nd \qquad\qquad F = m\ddot{x} = -kx$$

$$\downarrow$$

$$N = \text{total number of electrons} \qquad \boxed{\omega_p^2 = \frac{k}{m} = \frac{4\pi ne^2}{m}}$$

$$n = \frac{N}{V}$$

There is yet another way to derive the plasma frequency: go back to

$$\frac{\partial \vec{\jmath}}{\partial t} = \frac{ne^2}{m}\vec{E} - \frac{\vec{\jmath}}{\tau}$$

think of sloshing electrons
as producing a polarization current.

At high frequencies ($\omega\tau \gg 1$), $\omega \gg \frac{1}{\tau}$ we can neglect the last term. This leaves

$$\frac{\partial \vec{\jmath}}{\partial t} = \frac{ne^2}{m}\vec{E}$$

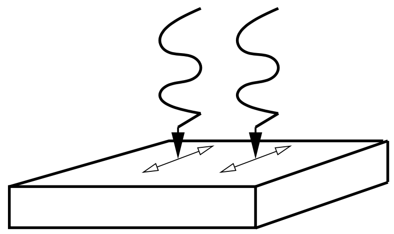
Recall the continuity equation: $\dfrac{\partial\rho}{\partial t} = -\nabla\cdot\vec{\jmath}$. So

$$\nabla\cdot\left[\frac{\partial\vec{\jmath}}{\partial t} = \frac{ne^2}{m}\vec{E}\right] \Rightarrow -\frac{\partial^2\rho}{\partial t^2} = \frac{ne^2}{m}\nabla\cdot\vec{E} = \frac{4\pi ne^2}{m}\rho$$

$$\text{or } \frac{\partial^2\rho}{\partial t^2} = -\omega_p^2\rho \qquad \text{where again } \omega_p^2 = \frac{4\pi ne^2}{m}$$

$$(m\ddot{x} = -kx\,\text{form})$$

Transverse EM Waves



If we shine EM radiation on a metal, it will not penetrate very far (and in fact, it will be reflected) for low frequencies because the electrons respond quickly enough to screen it. At high frequencies, however, ($\omega \gg \omega_p$) the electrons aren't fast enough to respond to $\vec{E}(t)$ and the radiation gets through. Thus the metals become transparent to ultraviolet light.

To see this mathematically, go back to Maxwell's eqns. and derive the wave equation.

$$\rho = 0 \Rightarrow \nabla\cdot\vec{E} = 0 \qquad\qquad \nabla\cdot\vec{B} = 0$$

$$\text{set } \mu = \varepsilon = 1 \qquad \nabla\times\vec{B} = \frac{4\pi}{c}\vec{\jmath} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \qquad (\vec{\jmath} = \sigma\vec{E})$$

$$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\times(\nabla\times\vec{E}) = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\vec{B})$$

$$\nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\frac{4\pi}{c}\sigma\vec{E} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t})$$

Fourier Transform w.r.t. time using $E \sim E_o e^{-i\omega t}$:

$$\begin{aligned}
\nabla^2 \vec{E} &= -\left(i\omega \frac{1}{c^2} 4\pi\sigma \vec{E} + \frac{\omega^2}{c^2} \vec{E} \right) \\
&= -\frac{\omega^2}{c^2} \left(\underbrace{1 + \frac{4\pi i \sigma}{\omega}}_{\varepsilon(\omega)} \right) \vec{E}
\end{aligned}$$

At low frequencies, $\omega\tau \ll 1$, $\sigma(\omega) \sim \sigma_0$ and $\frac{\omega^2}{c^2}$ term is negligible. Hence

$$\nabla^2 \vec{E} \simeq -i \frac{\omega}{c^2} 4\pi\sigma \vec{E}$$

For $\quad \vec{E} = \vec{E}_0 e^{i\vec{k}\cdot\vec{r}}, \quad \nabla^2 \vec{E} = -k^2 \vec{E} = -i \frac{\omega}{c^2} 4\pi\sigma \vec{E}$

$$\Rightarrow k^2 = i \frac{4\pi\sigma\omega}{c^2}$$

$$\Rightarrow k = \frac{\sqrt{4\pi\sigma\omega}}{c} \left(\frac{1+i}{\sqrt{2}} \right) = k' + ik''$$

$\vec{E} = \vec{E}_0 e^{ik'\cdot r} e^{-k''r} \leftarrow$ EM wave decays as it enters the metal

$$\text{Skin depth} = \frac{1}{k''} = \frac{c}{\sqrt{2\pi\sigma\omega}} \qquad (\omega\tau \ll 1)$$

High Frequencies

For $\omega\tau \gg 1, \quad \sigma(\omega) \sim \frac{\sigma_0}{-i\omega\tau} = i\frac{ne^2}{m\omega} \quad \left(\text{recall} \quad \sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau} \right)$

$$\begin{aligned}
\text{so} \quad \nabla^2 \vec{E} &= \frac{4\pi}{c^2} \frac{ne^2}{m} \vec{E} - \frac{\omega^2}{c^2} \vec{E} \qquad \omega_p^2 = \frac{4\pi n e^2}{m} \\
&= \frac{\omega_p^2 - \omega^2}{c^2} \vec{E} \quad \Rightarrow k^2 = -\frac{\omega_p^2 - \omega^2}{c^2}
\end{aligned}$$

For $\omega < \omega_p$, this leads to exponential decay with decay length $\frac{c}{\sqrt{\omega_p^2 - \omega^2}}$.

For $\omega > \omega_p$, we get propagation and the metal becomes transparent at a frequency $\nu_p \sim 10^{16}$ sec^{-1}.