LECTURE 4

Blackbody Radiation

Blackbody radiation is a nice example of the statistical mechanics that we have been discussing. A black body is a perfect absorber and absorbs all the radiation incident on it. If its temperature is kept constant, then the amount of power it radiates must equal the amount of power it absorbs. Otherwise it would heat up or cool off. We can imagine the black body being kept inside some kind of closed container which is at the same temperature T. The radiation field inside this enclosure is in equilibrium. In other words there is a gas of photons in thermal equilibrium inside the enclosure. By thermal equilibrium, we mean that the average occupation number $< n_s >$ of the single particle states is given by the Planck distribution that we talked about in lecture 3.

$$< n_s >= \frac{1}{e^{\beta\varepsilon_s} - 1} \tag{1}$$

One can imagine making a histogram by counting the photon energy density in each frequency range from ω to $\omega + d\omega$. It turns out that this distribution of the energy density of blackbody radiation is a universal curve that depends only on the temperature T. In other words if one plots the distribution of the photon energy density (counting both directions of polarization) as a function of photon (angular) frequency ω, the shape of the curve is universal and the position of the peak is a function only of the temperature. When we say that the curve is universal, we mean that it doesn't depend on the size or shape of the box, or what the walls are made of. All that matters is the temperature.

Blackbody radiation is historically important in physics for two reasons. The first is that the measurement of the spectral distribution in the late 1800's led Planck to come up with the idea of energy quantization. He couldn't explain the distribution unless he postulated that $E = h\nu$. This marked the birth of quantum mechanics. The second reason that blackbody radiation is important is that 3 K black body radiation pervades the universe and is the remnant of the Big Bang. This radiation is in the microwave region. We will come back to this at the end of the course.

Let's calculate the distribution of the mean energy density of blackbody radiation. Since the size and shape of the box don't matter, let's imagine a rectangular box of volume V filled with a gas of photons that are in thermal equilibrium. The box has edges with lengths L_x, L_y, and L_z such that each of these lengths is much larger than the longest wavelength of significance. There are 2 factors that determine the energy density at a given frequency. The first is the average energy in each state s which is given by

$$< n_s > \varepsilon_s = \frac{\varepsilon_s}{e^{\beta\varepsilon_s} - 1} \tag{2}$$

If we set $\varepsilon_s = \hbar\omega$ and $< n_s >=< n(\hbar\omega) >$, we can rewrite this to give:

$$< n(\hbar\omega) > \hbar\omega = \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} \tag{3}$$

The second factor is the number of states per unit volume whose frequency lies in the range between ω and $\omega + d\omega$. We figured this out at the end of lecture 1. We found that

$$N(\omega)d\omega = \frac{1}{\pi^2 c^3}\omega^2 d\omega \tag{4}$$

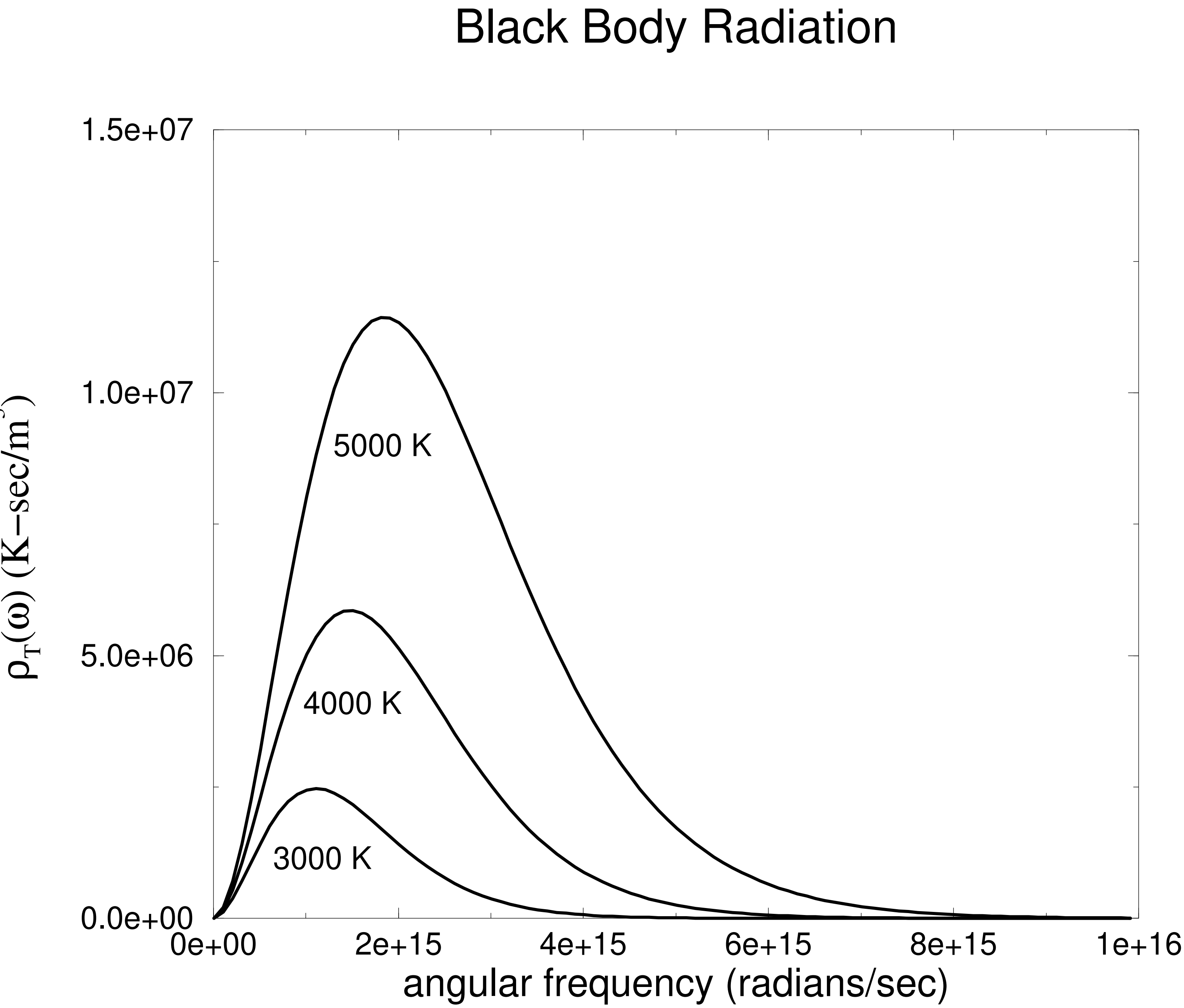
So at temperature T the mean energy density $\rho_T(\omega)d\omega$ contained in the photon gas by photons whose frequencies are between ω and $\omega + d\omega$ is given by the product of the average energy in each single photon state and the density of states which lie in this frequency range:

$$\rho_T(\omega)d\omega =< n(\hbar\omega) > \hbar\omega N(\omega)d\omega = \frac{\omega^2}{\pi^2 c^3}\frac{\hbar\omega}{(e^{\beta\hbar\omega} - 1)}d\omega \tag{5}$$

We can rewrite this to give:

$$\rho_T(\omega)d\omega = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{d\omega}{(e^{\beta\hbar\omega} - 1)} \tag{6}$$

This is Planck's law for the blackbody spectrum.



Black Body Radiation
1.5e+07
1.0e+07
5.0e+06
0.0e+00
ρT(ω) (K−sec/m^3)
5000 K
4000 K
3000 K
0e+00
2e+15
4e+15
6e+15
8e+15
1e+16
angular frequency (radians/sec)

We can take the high temperature limit to get the classical limit of this spectrum. In the high temperature limit, β is small so we can expand the exponential in the denominator:

$$e^{\beta\hbar\omega} - 1 \approx (1 + \beta\hbar\omega) - 1 = \beta\hbar\omega \tag{7}$$

So the high temperature limit of (6) is

$$\rho_T(\omega)d\omega \approx \frac{\hbar\omega^3}{\pi^2 c^3(\beta\hbar\omega)} d\omega \tag{8}$$

or, using $\beta = 1/k_B T$, we can write

$$\rho_T(\omega)d\omega \approx \frac{\omega^2 k_B T}{\pi^2 c^3} d\omega \tag{9}$$

This is the Rayleigh–Jeans formula for blackbody radiation. Notice that eqn. (9) increases as ω^2. Therefore the classical spectrum (9) predicts that the energy density goes to infinity as the frequency goes to infinity. By the end of the 1800's the black body spectrum had been measured and the classical formula had been calculated. There was a clear lack of agreement, so people knew they had a problem. Planck resolved the conflict by proposing that electromagnetic energy was not continuous, but rather was quantized. He proposed $E = \hbar\omega$ (or $E = h\nu$) and derived Planck's law (6). This fit the data very well, and quantum mechanics was born.

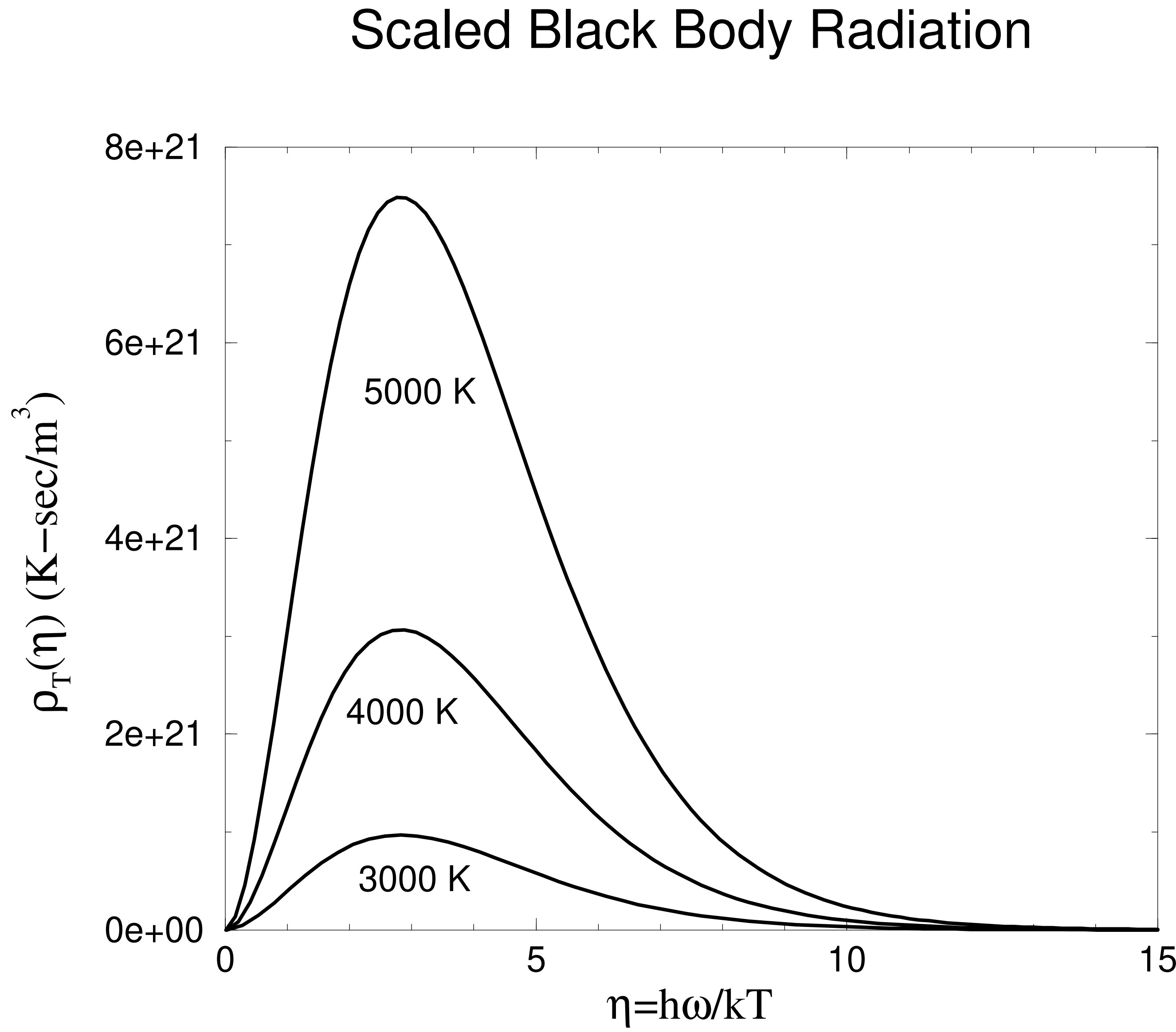
We can rewrite (6) in terms of a dimensionless parameter η:

$$\eta \equiv \beta\hbar\omega = \frac{\hbar\omega}{k_B T} \tag{10}$$

Planck's law becomes:

$$\rho_T(\eta)d\eta = \frac{\hbar}{\pi^2 c^3}\left(\frac{k_B T}{\hbar}\right)^4 \frac{\eta^3 d\eta}{e^\eta - 1} \tag{11}$$

If we plot $\rho_T(\eta)$ versus η, the maximum occurs around $\eta_{max} \approx 3$.



Scaled Black Body Radiation
8e+21
6e+21
4e+21
2e+21
0e+00
0
5
10
15
5000 K
4000 K
3000 K
ρT(η) (K−sec/m3)
η=hω/kT

So if at temperature T_1 the maximum occurs at frequency $\omega_{1,max}$, then at some other temperature T_2 the maximum occurs at $\omega_{2,max}$. This is because

$$\eta_{max} = \frac{\hbar\omega_{1,max}}{k_B T_1} = \frac{\hbar\omega_{2,max}}{k_B T_2} \tag{12}$$

or

$$\frac{\omega_{1,max}}{T_1} = \frac{\omega_{2,max}}{T_2} \tag{13}$$

This is called the Wien displacement law. It says that

$$\omega_{max} \propto T \tag{14}$$

This was initially an empirical relation that was deduced from the experimental data. We see that it also follows from Planck's law. It is often useful in physics to express things in terms of dimensionless parameters. The Wien displacement law is an example of useful scaling relations that can result from this.

We can also calculate the total energy density R_T contained in the photon gas at temperature T by integrating (6) over frequency:

$$R_T = \int_0^\infty \rho_T(\omega)d\omega \tag{15}$$

Using (11), we can rewrite this as

$$R_T = \frac{\hbar}{\pi^2 c^3}\left(\frac{k_B T}{\hbar}\right)^4 \int_0^\infty \frac{\eta^3 d\eta}{e^\eta - 1} \tag{16}$$

One can evaluate the integral exactly. The answer is

$$\int_0^\infty \frac{\eta^3 d\eta}{e^\eta - 1} = \frac{\pi^4}{15} \tag{17}$$

Using this, one finds

$$R_T = \frac{\pi^2}{15}\frac{(k_B T)^4}{(\hbar c)^3} \tag{18}$$

This is known as the Stefan–Boltzmann law. The important point is that the total energy density goes as the fourth power of the temperature:

$$R_T \propto T^4 \tag{19}$$

Finally the mean pressure $< p >$ exerted on the walls of the enclosure by the radiation is simply related to the total energy density:

$$< p >= \sum_s < n_s > \left(-\frac{\partial \varepsilon_s}{\partial V}\right) = \frac{1}{3}R_T \tag{20}$$

(The pressure can also be written as $\beta^{-1}\left(\frac{\partial \ln Z}{\partial V}\right) = -\frac{\partial F}{\partial V}$.) The "3" in the denominator reflects the fact that the box is 3 dimensional. Radiation pressure is quite small, but it is what gives comets their tails. Solar radiation is what pushes tiny bits of dust and ice that come from the ice ball away from the sun and produces the tail. The comet tail always points away from the sun. We will see examples later where a microscopic understanding of radiation pressure leads to laser cooling of beams of atoms.

Atomic Orbitals

It's always a good idea to have some idea of how big things are so you can easily estimate what's important at a particular energy or length scale. Visible light consists of radiation with wavelengths in the range of 4000 Å(violet) to 7500 Å(red). When ordinary white light passes through a prism, it breaks up into a continuous rainbow spectrum. But if you take a bead of sodium, for example, and put it in a flame, it will give off yellow light. If you pass this light through a prism, you get a spectrum of distinct lines, not a continuous spectrum. Each line represents a definite wavelength or energy. Each element has its own characteristic line spectrum; it's like a fingerprint. These lines are a result of the fact that energy levels in atoms are quantized. So electrons in the sodium atoms get excited into higher energy levels by the flame. When these excited electrons make transitions to lower energy levels they emit photons in order to conserve energy. These photons produce the line spectrum. A famous set of lines is the Balmer series which is emitted by hydrogen in the visible region of the spectrum. It was found empirically that the frequencies of these lines can be expressed by the formula

$$\nu = 3.29 \times 10^{15}\ \text{Hz}\left(\frac{1}{2^2} - \frac{1}{n^2}\right) \tag{21}$$

where n is an integer equal to, or greater than, 3.

To explain the Balmer series Bohr proposed that the electron of hydrogen can exist only in certain spherical orbits (called energy levels or shells), which are arranged concentrically around the nucleus. These orbits are subject to a quantum restriction: the orbital angular momentum is quantized in units of $\hbar$:

$$mvr = n\hbar \tag{22}$$

where $n = 1, 2, 3$ ... Here m is the mass of the electron, v is its velocity, and r is the radius of the orbit. Each orbit has a characteristic energy. As long as the electron remains in a given orbit, it neither absorbs nor radiates energy. Thus, the K level ($n = 1$), the shell closest to the nucleus, has the smallest radius and lowest energy. The next shell (L; $n = 2$) has a larger radius and a higher energy. The shells are labelled K ($n = 1$), L ($n = 2$), M ($n = 3$), N ($n = 4$), and O ($n = 5$), in order of increasing radius from the nucleus.

We can deduce the energy of the different n levels using Bohr's postulate in the following way. Setting the centripetal force equal to the Coulomb force on an electron from the nucleus, we obtain:

$$\frac{mv^2}{r} = \frac{Ze^2}{r^2} \tag{23}$$

where Z is the number of protons in the nucleus. Z is called the atomic number. We can rewrite this:

$$v^2 = \frac{Ze^2}{mr} \tag{24}$$

Now we use Bohr's quantization condition:

$$\begin{aligned} mvr &= n\hbar \\ v &= \frac{n\hbar}{mr} \end{aligned} \tag{25}$$

$$\tag{26}$$

Squaring (26) and inserting this into (24), we get

$$\frac{Ze^2}{mr} = \frac{n^2\hbar^2}{m^2r^2} \tag{27}$$

or

$$r = \frac{n^2\hbar^2}{mZe^2} \tag{28}$$

If we solve this for the smallest orbit ($n = 1$) of hydrogen ($Z = 1$), we get the Bohr radius r_o:

$$r_o = \frac{\hbar^2}{me^2} \tag{29}$$

Numerically $r_o = 0.529$ Å. This is a good number to remember since it is a characteristic atomic size.

The total energy of the electron is the sum of its kinetic plus its potential energy:

$$E = \frac{1}{2}mv^2 - \frac{Ze^2}{r} \tag{30}$$

Using (23), we have

$$mv^2 = \frac{Ze^2}{r} \tag{31}$$

Therefore,

$$E = \frac{Ze^2}{2r} - \frac{Ze^2}{r} = -\frac{Ze^2}{2r} \tag{32}$$

Substituting in (28), we get

$$E_n = -\frac{mZ^2e^4}{2n^2\hbar^2} \tag{33}$$

Here we see that the quantization of momentum leads to the quantization of the energy of the Bohr orbits.

When the electrons are arranged as close as possible to the nucleus (in the case of hydrogen, one electron in the K level), they are in the lowest energy state which is called the ground state. When an atom absorbs energy, one or more of the electrons can jump to higher energy levels, leaving the atom in an excited state. When an electron falls back to a lower orbit, it emits an definite amount of energy equal to the difference in energy between the two levels. Let E_o denote the energy of the initial outer level and E_i be the energy of the final inner level. This energy is emitted as a photon with a characteristic energy. The energy of the photon is

$$h\nu = E_o - E_i \tag{34}$$

Substituting in E_n from (33), we get

$$h\nu = \frac{mZ^2e^4}{2\hbar^2}\left(\frac{1}{n_i^2} - \frac{1}{n_o^2}\right) \tag{35}$$

or

$$\nu = \frac{mZ^2e^4}{4\pi\hbar^3}\left(\frac{1}{n_i^2} - \frac{1}{n_o^2}\right) \tag{36}$$

If we evaluate this expression for hydrogen with $Z = 1$, the frequencies of photons emitted by transitions from higher levels n to the $n = 2$ level leads to the Balmer series formula

$$\nu = 3.29 \times 10^{15}\ \mathrm{Hz}\left(\frac{1}{2^2} - \frac{1}{n^2}\right) \tag{37}$$

Photons emitted due to electronic transitions to the $n = 1$ level are in the ultraviolet region. These transitions give rise to the Lyman series. The lines associated with having an $n = 3$ final state are in the infrared and are called the Paschen series.

If an electron is removed to an infinite shell, the atom is said to be ionized; the ionization energy is the minimum amount of energy required for this process.