PROBLEM SET 6

Nov. 7 Colloquium: “Linear Collider”

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3:30 pm, 101 Rowland Hall

1. (20 pts) In the Weiss (mean-field) theory of ferromagnetism the Gibbs free energy

\[ G = E(M) - HM - TS, \]

where \( H \) is the externally applied magnetic field, \( M \) is the magnetization, \( E \) is the internal energy, and \( S \) is the entropy) has the form

\[ G = G_o(T) + a(T)M^2 + b(T)M^4 + O(M^6) - MH \]

where \( G_o(T) \) is independent of \( M \), and where the coefficient \( b(T) \) is a slowly varying function of \( T \) but \( a(T) \) is of the form \( a_o(T - T_C) \), \( T_C \) being the critical temperature in Weiss theory. Assume that \( T \) is close to \( T_C \) and that \( M \) is small. Using the fact that in thermal equilibrium \( M \) will take the value which minimizes \( G \), find (a) the equilibrium value of \( M \) for \( H = 0 \), for \( T > T_C \) and \( T < T_C \); (b) the form of \( M \) at \( T_C \) as a function of \( H \); (c) the zero—field differential susceptibility \( \chi = (\partial M/\partial H)_{T, H=0} \) for \( T > T_C \) and \( T < T_C \); and (d) the discontinuity in the specific heat at constant \( H \)

\( C_H = -T(\partial^2 G/\partial T^2)H \)

at the point \( T = T_C \) and \( H = 0 \). In other words find the difference between the limits of \( C_H(T, H = 0) \) as \( T \to T_C \) from above and from below.

2. Meissner Effect In deriving flux quantization in a superconductor, we found that the electric current is given by

\[ \tilde{j} = q\psi^*\tilde{\psi} = \frac{q\psi}{m} \left( \hbar \nabla \theta - \frac{q}{c} \tilde{A} \right) \]

(a) Use this and the appropriate Maxwell equation to show that

\[ \nabla^2 \tilde{B} = \lambda^{-2} \tilde{B} \]
What is \( \lambda \) in terms of the density of Cooper pairs \( n_p, \epsilon \), the mass of the electron \( m \), and \( \epsilon^2 \)? \( \lambda \) is called the London penetration depth. (Hint: Use some vector identities to simplify the equations. See inside cover of Jackson’s *Classical Electrodynamics*, for example.)

(b) Suppose that \( \vec{B} \) points along the \( z \) axis and only varies in the \( x \) direction. Suppose the superconductor fills the half space \( x > 0 \) and there is vacuum for \( x < 0 \). Show that \( B_x \) dies out exponentially as it penetrates the superconductor in the \( x \) direction. (Don’t worry about the prefactor of the exponential.) In other words the magnetic field dies out exponentially as you go into the superconductor. This is the Meissner effect.

3. **AC Josephson Effect** When a static DC voltage \( V \) is applied across a Josephson junction, an AC current results. To see how this comes about, notice that an electron pair experiences a potential energy difference \( qV \) on passing across the junction, where \( q = -2e \). We can say that a pair on one side is at potential \(-eV\) and a pair on the other side is at \(+eV\). Thus the equations of motion become

\[
i\hbar \frac{\partial \psi_1}{\partial t} = \hbar T \psi_2 - eV \psi_1 \quad i\hbar \frac{\partial \psi_2}{\partial t} = \hbar T \psi_1 + eV \psi_2
\]

where \( \psi_1 \) is the superconducting order parameter on side 1:

\[
\psi_1 = \sqrt{n_1} e^{i\theta_1}
\]

\( n_1 \) is the density of superconducting pairs on side 1. Similarly

\[
\psi_2 = \sqrt{n_2} e^{i\theta_2}
\]

Assume that the superconductors are identical. Find the current density \( J \) as a function of time and of the phase difference \( \delta(0) \). \( \delta(0) = \theta_2 - \theta_1 \) is the phase difference at \( V = 0 \). What is the angular frequency \( \omega \) at which the current oscillates when a voltage \( V \) is applied?