(*a*) if N is even?

(*b*) if N is odd?

1.5 In the game of Russian roulette (*not* recommended by the author), one inserts a single cartridge into the drum of a revolver, leaving the other five chambers of the drum empty. One then spins the drum, aims at one's head, and pulls the trigger.

(*a*) What is the probability of being still alive after playing the game N times?

(*b*) What is the probability of surviving $(N - 1)$ turns in this game and then being shot the Nth time one pulls the trigger?

(*c*) What is the mean number of times a player gets the opportunity of pulling the trigger in this macabre game?

1.6 Consider the random walk problem with $p = q$ and let $m = n_1 - n_2$ denote the net displacement to the right. After a total of N steps, calculate the following mean values: $\overline{m}$, $\overline{m^2}$, $\overline{m^3}$, and $\overline{m^4}$.

1.7 Derive the binomial distribution in the following algebraic way, which does not involve any explicit combinatorial analysis. One is again interested in finding the probability $W(n)$ of n successes out of a total of N independent trials. Let $w_1 \equiv p$ denote the probability of a success, $w_2 = 1 - p = q$ the corresponding probability of a failure. Then $W(n)$ can be obtained by writing

$$W(n) = \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} \cdots \sum_{m=1}^{2} w_i w_j w_k \cdots w_m \tag{1}$$

Here each term contains N factors and is the probability of a particular combination of successes and failures. The sum over all combinations is then to be taken only over those terms involving w_1 exactly n times, i.e., only over those terms involving $w_1{}^n$.

By rearranging the sum (1), show that the *un*restricted sum can be written in the form

$$W(n) = (w_1 + w_2)^N$$

Expanding this by the binomial theorem, show that the sum of all terms in (1) involving $w_1{}^n$, i.e., the desired probability $W(n)$, is then simply given by the one binomial expansion term which involves $w_1{}^n$.

1.8 Two drunks start out together at the origin, each having equal probability of making a step to the left or right along the x axis. Find the probability that they meet again after N steps. It is to be understood that the men make their steps simultaneously. (It may be helpful to consider their relative motion.)

1.9 The probability $W(n)$ that an event characterized by a probability p occurs n times in N trials was shown to be given by the binomial distribution

$$W(n) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n} \tag{1}$$

Consider a situation where the probability p is small ($p \ll 1$) and where one is interested in the case $n \ll N$. (Note that if N is large, $W(n)$ becomes very small if $n \to N$ because of the smallness of the factor p^n when $p \ll 1$. Hence $W(n)$ is indeed only appreciable when $n \ll N$.) Several approximations can then be made to reduce (1) to simpler form.

(*a*) Using the result $\ln(1-p) \approx -p$, show that $(1-p)^{N-n} \approx e^{-Np}$.
(*b*) Show that $N!/(N-n)! \approx N^n$.
(*c*) Hence show that (1) reduces to

$$W(n) = \frac{\lambda^n}{n!} e^{-\lambda} \tag{2}$$

where $\lambda \equiv Np$ is the mean number of events. The distribution (2) is called the "Poisson distribution."

1.10 Consider the Poisson distribution of the preceding problem.

(*a*) Show that it is properly normalized in the sense that $\sum_{n=0}^{N} W_n = 1$. (The sum can be extended to infinity to an excellent approximation, since W_n is negligibly small when $n \gtrsim N$.)

(*b*) Use the Poisson distribution to calculate $\bar{n}$.

(*c*) Use the Poisson distribution to calculate $\overline{(\Delta n)^2} \equiv \overline{(n - \bar{n})^2}$.

1.11 Assume that typographical errors committed by a typesetter occur completely at random. Suppose that a book of 600 pages contains 600 such errors. Use the Poisson distribution to calculate the probability

(*a*) that a page contains no errors
(*b*) that a page contains at least three errors

1.12 Consider the α particles emitted by a radioactive source during some time interval t. One can imagine this time interval to be subdivided into many small intervals of length Δt. Since the α particles are emitted at random times, the probability of a radioactive disintegration occurring during any such time Δt is completely independent of whatever disintegrations occur at other times. Furthermore, Δt can be imagined to be chosen small enough so that the probability of more than one disintegration occurring in a time Δt is negligibly small. This means that there is some probability p of one disintegration occurring during a time Δt (with $p \ll 1$, since Δt was chosen small enough) and probability $1 - p$ of no disintegration occurring during this time. Each such time interval Δt can then be regarded as an independent trial, there being a total of $N = t/\Delta t$ such trials during a time t.

(*a*) Show that the probability $W(n)$ of n disintegrations occurring in a time t is given by a Poisson distribution.

(*b*) Suppose that the strength of the radioactive source is such that the mean number of disintegrations per minute is 24. What is the probability of obtaining n counts in a time interval of 10 seconds? Obtain numerical values for all integral values of n from 0 to 8.

1.13 A metal is evaporated in vacuum from a hot filament. The resultant metal atoms are incident upon a quartz plate some distance away and form there a thin metallic film. This quartz plate is maintained at a low temperature so that any metal atom incident upon it sticks at its place of impact without further migration. The metal atoms can be assumed equally likely to impinge upon any element of area of the plate.

If one considers an element of substrate area of size b^2 (where b is the metal atom diameter), show that the number of metal atoms piled up on this area should be distributed approximately according to a Poisson distribution. Suppose that one evaporates enough metal to form a film of mean thickness corresponding to 6 atomic layers. What fraction of the substrate area is then not covered by metal at all? What fraction is covered, respectively, by metal layers 3 atoms thick and 6 atoms thick?

1.14 A penny is tossed 400 times. Find the probability of getting 215 heads. (Suggestion: use the Gaussian approximation.)

1.15 A set of telephone lines is to be installed so as to connect town A to town B. The town A has 2000 telephones. If each of the telephone users of A were to be guaranteed instant access to make calls to B, 2000 telephone lines would be needed. This would be rather extravagant. Suppose that during the busiest hour of the day each subscriber in A requires, on the average, a telephone connection to B for two minutes, and that these telephone calls are made at random. Find the minimum number M of telephone lines to B which must be installed so that at most only 1 percent of the callers of town A will fail to have immediate access to a telephone line to B. (Suggestion: approximate the distribution by a Gaussian distribution to facilitate the arithmetic.)

1.16 Consider a gas of N_0 noninteracting molecules enclosed in a container of volume V_0. Focus attention on any subvolume V of this container and denote by N the number of molecules located within this subvolume. Each molecule is equally likely to be located anywhere within the container; hence the probability that a given molecule is located within the subvolume V is simply equal to V/V_0.

(*a*) What is the mean number $\bar{N}$ of molecules located within V? Express your answer in terms N_0, V_0, and V.

(*b*) Find the relative dispersion $\overline{(N - \bar{N})^2}/\bar{N}^2$ in the number of molecules located within V. Express your answer in terms of $\bar{N}$, V, and V_0.

(*c*) What does the answer to part (*b*) become when $V \ll V_0$?

(*d*) What value should the dispersion $\overline{(N - \bar{N})^2}$ assume when $V \to V_0$? Does the answer to part (*b*) agree with this expectation?

1.17 Suppose that in the preceding problem the volume V under consideration is such that $0 \ll V/V_0 \ll 1$. What is the probability that the number of molecules in this volume is between N and $N + dN$?

1.18 A molecule in a gas moves equal distances l between collisions with equal probability in any direction. After a total of N such displacements, what is the mean square displacement $\overline{R^2}$ of the molecule from its starting point?

1.19 A battery of total emf V is connected to a resistor R; as a result an amount of power $P = V^2/R$ is dissipated in this resistor. The battery itself consists of N individual cells connected in series so that V is just equal to the sum of the emf's of all these cells. The battery is old, however, so that not all cells are in perfect condition. Thus there is only a probability p that the emf of any individual cell has its normal value v; and a probability $1 - p$ that the emf of any individual cell is zero because the cell has become internally shorted. The individual cells are statistically independent of each other. Under these conditions, calculate the *mean* power $\bar{P}$ dissipated in the resistor, expressing the result in terms of N, v, and p.

1.20 Consider N similar antennas emitting linearly polarized electromagnetic radiation of wavelength λ and velocity c. The antennas are located along the x axis at a separation λ from each other. An observer is located on the x axis at a great distance from the antennas. When a *single* antenna radiates, the observer measures an *intensity* (i.e., mean-square electric-field amplitude) equal to I.

(*a*) If all the antennas are driven in phase by the same generator of frequency $\nu = c/\lambda$, what is the total intensity measured by the observer?

by which the known weight descends. By doing such work the system can be brought from its initial macrostate of volume V_i and pressure $\bar{p}_i$ to a final state of volume V_f and pressure $\bar{p}_f$. But this can be done in many ways: e.g., by rotating the paddle wheel first and then moving the piston the required amount; or by moving the piston first and then rotating the paddle wheel through the requisite number of revolutions; or by performing these two types of work alternately in smaller amounts. The statement $(2 \cdot 11 \cdot 9)$ asserts that if the *total* work performed in each such procedure is measured, the result is always the same.*

Similarly, it follows that if the external parameters of a system are kept fixed so that it does no work, then $đW = 0$ and $(2 \cdot 8 \cdot 3)$ reduces to

$$đQ = d\bar{E}$$

so that $đQ$ becomes an exact differential. The amount of heat Q absorbed in going from one macrostate to another is then independent of the process used and depends only on the mean energy difference between them.

SUGGESTIONS FOR SUPPLEMENTARY READING

Statistical formulation

R. C. Tolman: "The Principles of Statistical Mechanics," chaps. 3 and 9, Oxford University Press, Oxford, 1938. (This book is a classic in the field of statistical mechanics and is entirely devoted to a careful exposition of fundamental ideas. The chapters cited discuss ensembles of systems and the fundamental statistical postulate in classical and quantum mechanics, respectively.)

Work and heat—macroscopic discussion

M. W. Zemansky: "Heat and Thermodynamics," 4th ed, chaps. 3 and 4, McGraw-Hill Book Company, New York, 1957.

H. B. Callen: "Thermodynamics," secs. 1.1–1.7, John Wiley & Sons, Inc., New York, 1960. (The analogy mentioned on pp. 19 and 20 is particularly instructive.)

PROBLEMS

2.1 A particle of mass m is free to move in one dimension. Denote its position coordinate by x and its momentum by p. Suppose that this particle is confined within a box so as to be located between $x = 0$ and $x = L$, and suppose that its energy is known to lie between E and $E + \delta E$. Draw the classical phase space

* Paddle wheels such as this were historically used by Joule in the last century to establish the equivalence of heat and mechanical energy. In the experiment just mentioned we might equally well replace the paddle wheel by an electric resistor on which electrical work can be done by sending through it a known electric current.

of this particle, indicating the regions of this space which are accessible to the particle.

2.2 Consider a system consisting of two weakly interacting particles, each of mass m and free to move in one dimension. Denote the respective position coordinates of the two particles by x_1 and x_2, their respective momenta by p_1 and p_2. The particles are confined within a box with end walls located at $x = 0$ and $x = L$. The total energy of the system is known to lie between E and $E + \delta E$. Since it is difficult to draw a four-dimensional phase space, draw separately the part of the phase space involving x_1 and x_2 and that involving p_1 and p_2. Indicate on these diagrams the regions of phase space accessible to the system.

2.3 Consider an ensemble of classical one-dimensional harmonic oscillators.

(*a*) Let the displacement x of an oscillator as a function of time t be given by $x = A \cos (\omega t + \varphi)$. Assume that the phase angle φ is equally likely to assume any value in its range $0 < \varphi < 2\pi$. The probability $w(\varphi)\, d\varphi$ that φ lies in the range between φ and $\varphi + d\varphi$ is then simply $w(\varphi)\, d\varphi = (2\pi)^{-1}\, d\varphi$. For any fixed time t, find the probability $P(x)\, dx$ that x lies between x and $x + dx$ by summing $w(\varphi)\, d\varphi$ over all angles φ for which x lies in this range. Express $P(x)$ in terms of A and x.

(*b*) Consider the classical phase space for such an ensemble of oscillators, their energy being known to lie in the small range between E and $E + \delta E$. Calculate $P(x)\, dx$ by taking the ratio of that volume of phase space lying in this energy range *and* in the range between x and $x + dx$ to the total volume of phase space lying in the energy range between E and $E + \delta E$ (see Fig. 2·3·1). Express $P(x)$ in terms of E and x. By relating E to the amplitude A, show that the result is the same as that obtained in part (*a*).

2.4 Consider an isolated system consisting of a large number N of very weakly interacting localized particles of spin $\frac{1}{2}$. Each particle has a magnetic moment μ which can point either parallel or antiparallel to an applied field H. The energy E of the system is then $E = -(n_1 - n_2)\mu H$, where n_1 is the number of spins aligned parallel to H and n_2 the number of spins aligned antiparallel to H.

(*a*) Consider the energy range between E and $E + \delta E$ where δE is very small compared to E but is microscopically large so that $\delta E \gg \mu H$. What is the total number of states $\Omega(E)$ lying in this energy range?

(*b*) Write down an expression for $\ln \Omega(E)$ as a function of E. Simplify this expression by applying Stirling's formula in its simplest form (A·6·2).

(*c*) Assume that the energy E is in a region where $\Omega(E)$ is appreciable, i.e., that it is not close to the extreme possible values $\pm N\mu H$ which it can assume. In this case apply a Gaussian approximation to part (*a*) to obtain a simple expression for $\Omega(E)$ as a function of E.

2.5 Consider the infinitesimal quantity

$$A\, dx + B\, dy \equiv đF$$

where A and B are both functions of x and y.

(*a*) Suppose that $đF$ is an exact differential so that $F = F(x,y)$. Show that A and B must then satisfy the condition

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$$

(*b*) If $đF$ is an exact differential, show that the integral $\int đF$ evaluated along *any* closed path in the xy plane must vanish.