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PHYSICS 214A
Final Exam Solutions
March 19, 2009

1. Do not open the exam until instructed to do so.
2. Write your answers in the spaces provided for each part of each problem. Place answers in the boxes if provided. Show your calculations in the available space or the blank facing page. Please write clearly. If we can't read your writing, you won't get credit.
3. Clearly explain your solutions. Give explanations in words for the main steps of your arguments and define your symbols and variables.

$R = 8.314$ J/K-mol	$k_B = 1.38054 \times 10^{-23}$ J/K
1 atm $= 1.013 \times 10^5$ N/m^2	273 K $= 0^o$ C
$\hbar = 1.05 \times 10^{-34}$ J s	$c = 3.00 \times 10^8$ m/s

DO NOT WRITE BELOW THIS LINE

A	MAX POINTS	SCORE	INITIALS
Problem 1	60		
Problem 2	10		
Problem 3	10		
Problem 4	10		
Problem 5	10		
Total	100		

1. **Short Answer**(60 points)

   (a) (6 points) A block of silicon has a heat capacity of 1 J/K at $T = 20$ K. Estimate the heat capacity C at $T = 10$ K. Silicon is an insulator at these temperatures.

   Answer: Use the Debye specific heat which has the form

$$C_V = AT^3 \tag{1}$$

   where A is a constant that we can find as follows:

$$A = \frac{C_V(T = 20 \text{ K})}{(20 \text{ K})^3} \tag{2}$$

   So

$$\begin{aligned} C_V(T = 10 \text{ K}) &= \frac{(10 \text{ K})^3}{(20 \text{ K})^3} C_V(T = 20 \text{ K}) \\ &= \frac{1}{8} C_V(T = 20 \text{ K}) \\ &= 0.125 \text{ J/K} \end{aligned} \tag{3}$$

   $C = 0.125$ J/K

   (b) (6 points) The electrical resistivity ρ of a metal at room temperature is proportional to the probability that an electron is scattered by the vibrating atoms in the lattice, and this probability is in turn proportional to the mean square amplitude of vibration of these atoms. Assuming classical statistics to be valid in this temperature range, what is the dependence of the electrical resistivity ρ on the absolute temperature T?

   Answer: From the equipartition theorem,

$$\langle \frac{1}{2} m\omega^2 x^2 \rangle = \frac{3}{2} k_B T \tag{4}$$

   So

$$\rho \sim \langle x^2 \rangle \sim T \tag{5}$$

   $\rho(T) \sim T$

(c) (6 points) A wire of length L, radius r, and resistance R carries a current I. If thermal conduction along the wire is negligible, write an equation for the steady state temperature T of the wire. Explain your reasoning.

Answer: The power $P = I^2R$ dissipated in the wire equals the power radiated by blackbody radiation P_r where

$$P_r = \mathcal{P}_e^{tot} A \tag{6}$$

where the total power $\mathcal{P}_e^{tot}$ emitted per unit area is given by the Stefan-Boltzmann law:

$$\mathcal{P}_e^{tot} = \sigma T^4 \qquad \sigma \equiv \frac{\pi^2}{60}\frac{k_B^4}{c^2\hbar^3} \tag{7}$$

and A is the surface area of the wire:

$$A = 2\pi r L \tag{8}$$

So we have

$$\begin{aligned} P &= \sigma T^4 A = I^2 R \\ T^4 &= \frac{I^2R}{\sigma A} = \frac{I^2R}{2\pi r\sigma L} \end{aligned} \tag{9}$$

$$\boxed{T = \left(\frac{I^2R}{2\pi r\sigma L}\right)^{1/4}}$$

(d) (6 points) In the pressure versus temperature phase diagram for ^{3}He, the melting curve has a negative slope ($dp/dT < 0$). Like most substances, solid ^{3}He is denser than liquid ^{3}He. Suppose we have an equilibrium mixture of solid and liquid ^{3}He. If a small amount of heat Q is added to this mixture, will some solid converted into liquid or will some liquid be converted into solid? Or neither? Assume the mixture is still a mixture of solid and liquid after the heat is added (though with possibly different proportions of liquid and solid). Explain your reasoning.

Answer: Liquid ^{3}He will be converted into solid. To understand why, note that from the Clausius-Clapeyron equation:

$$\frac{dp}{dT} = \frac{\Delta s}{\Delta v} < 0 \tag{10}$$

where s is the entropy per mole and v is the volume of a mole. Since the solid is denser than the liquid, the volume of 1 mole of the solid is less than the volume of 1 mole of the liquid. So

$$\Delta v = v_{\text{solid}} - v_{\text{liquid}} < 0 \tag{11}$$

Thus the Clausius-Clapeyron equation implies that

$$\begin{aligned} \Delta s &= s_{\text{solid}} - s_{\text{liquid}} > 0 \\ s_{\text{solid}} &> s_{\text{liquid}} \end{aligned}$$

So when heat Q is added, the entropy of the mixture increases by an amount

$$\Delta S = \frac{Q}{T} \tag{12}$$

This added entropy goes into converting liquid into solid which is the higher entropy phase. Note that when solid and liquid coexist, the temperature T stays constant as long as the pressure P is constant.

Note that $\Delta S = Q/T$ is *not* the same Δs as one sees in the Clausius-Claperyon equation where $\Delta s = s_{\text{phase 1}} - s_{\text{phase 2}}$. For example, suppose

$$\begin{aligned} \frac{dp}{dT} &= \frac{\Delta s}{\Delta v} < 0 \\ \Delta v &= v_{\text{liquid}} - v_{\text{solid}} > 0 \\ \Delta s &= s_{\text{liquid}} - s_{\text{solid}} < 0 \end{aligned}$$

So when heat is added, $Q = T\Delta S > 0$, but we can have $\Delta s = s_{\text{liquid}} - s_{\text{solid}} < 0$, and $s_{\text{liquid}} < s_{\text{solid}}$.

(e) (6 points)

i. Is a neutral ^{4}He atom a boson or a fermion? Explain your reasoning.
Answer: ^{4}He is a boson. A neutral atom has 2 protons, 2 neutrons, and 2 electrons. Adding the spin of 6 fermions, each with spin 1/2, gives an integer spin to ^{4}He. So ^{4}He is a boson because bosons have integer spin.

ii. Is a neutral ^{3}He atom a boson or a fermion? Explain your reasoning.
Answer: ^{3}He is a fermion. A neutral ^{3}He atom has 2 protons, 1 neutron, and 2 electrons. Adding the spin of 5 fermions, each with spin 1/2, yields a half integer spin. So ^{3}He is a fermion because fermions have half-integer spin.

(f) (6 points) State the ergodic hypothesis.

Answer: The ergodic hypothesis assumes that a system will visit all possible microstates after a sufficiently long time. It is assumed that the system traverses all the possible microstates fast enough that the time averages are identical with the averages taken over a large collection of identical and independent systems, i.e., an ensemble average.

(g) (6 points) What is an ensemble? What are the different types of ensembles? What quantities are held fixed for each of them? Be sure to define all variables.

Answer: An ensemble is a hypothetical collection of identically prepared systems, all subject to the same initial conditions and the same set $\{X\}$ of external parameters such as the total energy, particle number, and volume. The systems in the ensemble will, in general, be in different microstates and will, therefore, also be characterized by different macroscopic parameters (e.g., by different values of pressure or magnetization).

There are 3 different types of ensembles:

i. Microcanonical Ensemble fixes E and N where E is the total energy and N is the total number of particles.

ii. Canonical Ensemble fixes the temperature T and N.

iii. Grand Canonical Ensemble fixes T and the chemical potential μ.

(h) (6 points) Explain why water molecules in equilibrium form ice at low temperatures and liquid water at higher temperatures.

Answer: Ice has a lower internal energy E while water has a higher entropy S. The goal is to minimize the free energy $F = E - TS$ (or $G = E - TS + pV$). At high temperatures, TS is important which implies that large entropy is important. This, in turn, implies that water has a lower free energy than ice. At low temperatures, TS is small, so $F \approx E$ is lower for ice than for water.

(i) (6 points) What is the difference between intensive and extensive parameters? Give an example of each.

Answer: An extensive parameter is proportional to the size of the system. Examples of extensive parameters include the total mass, total volume, internal energy E, total entropy S, and heat capacity. An intensive parameter is unchanged if the size or the mass or the number of particles doubles. Examples of intensive parameters include temperature, pressure, and specific heat.

(j) (6 points) N noninteracting particles of mass m are in a box of volume V. The particles are in equilibrium with a heat bath at temperature T. In terms of these variables and the fundamental constants, what is the condition for the validity of the classical appoximation for calculating the thermodynamic properties of the gas? Show your work and explain your reasoning.

Answer: (See notes from lecture 11.) Let $\overline{\lambda}$ be the mean de Broglie wavelength. Let $\overline{R}$ be the mean separation of the ideal gas atoms. The classical approximation applies when

$$\overline{R} \gg \overline{\lambda} \tag{13}$$

i.e., the mean separation between particles is much greater than their mean de Broglie wavelength. One way to see this is to begin with the uncertainty relation:

$$\Delta x \Delta p \gtrsim \hbar \tag{14}$$

We expect $\Delta x \lesssim \overline{R}$ and $\Delta p \lesssim \overline{p}$ where $\overline{p}$ is the mean momentum. Then we would expect that classical mechanics would adequately describe the dynamics of the gas whenever

$$\overline{R}\overline{p} \gg \hbar \tag{15}$$

This effectively says that $\hbar$ is negligible. Recall that

$$\overline{p} = \frac{2\pi\hbar}{\overline{\lambda}} = \frac{h}{\overline{\lambda}} \tag{16}$$

Then

$$\overline{R}\overline{p} \gg \hbar \tag{17}$$

becomes

$$\overline{R} \gg \frac{\hbar}{\overline{p}} = \frac{\overline{\lambda}}{2\pi} \tag{18}$$

This is equivalent to Eq. (13).

To relate $\overline{R}$ and $\overline{\lambda}$ to the variables given in the problem, we note the following. First imagine each molecule at the center of a little cube of side $\overline{R}$, these cubes filling the available volume V. Then

$$\overline{R}^3 N = V \tag{19}$$

or

$$\overline{R} = \left(\frac{V}{N}\right)^{1/3} \tag{20}$$

Next, we use Eq. (16) to relate the mean de Broglie wavelength to the mean momentum:

$$\overline{p} = \frac{2\pi\hbar}{\overline{\lambda}} = \frac{h}{\overline{\lambda}} \tag{21}$$

The mean momentum $\overline{p}$ can be estimated using the equipartition theorem

$$\frac{\overline{p}^2}{2m} \approx \frac{3}{2}k_BT \tag{22}$$

or

$$\overline{p} \approx \sqrt{3mk_BT} \tag{23}$$

and

$$\overline{\lambda} \approx \frac{h}{\sqrt{3mk_BT}} \tag{24}$$

Hence the condition that the classical description should be valid is

$$\left(\frac{V}{N}\right)^{1/3} \gg \frac{h}{\sqrt{3mk_BT}} \tag{25}$$

So for large masses and high temperatures we expect classical mechanics to hold.

2. (10 points) A MOSFET (Metal-Oxide-Semiconductor Field-Effect Transistor) is commonly used to study two dimensional electron gases. A simple approximation to a MOSFET is a parallel plate capacitor with capacitance C. Each plate has an area A. Suppose a (gate) voltage V is applied across the capacitor plates. The charge on the capacitor is Q. Regard the charge $-Q$ as residing on the inner surface of the negatively charged plate so it constitutes a two dimensional electron gas. Find the Fermi energy E_F as a function of the voltage V.

   Solution: Let N be the number of electrons on the lower plate and e be the electron charge. Let N_o be the number of electrons on the lower plate when the voltage $V = 0$, and ΔN be the number of electrons added to the lower plate as a result of $V \neq 0$. So $N = N_o + \Delta N$. Then

$$-Q = -\Delta N e \qquad \text{or } Q = \Delta N e \tag{26}$$

   Find the density of states assuming isotropy:

$$\frac{A}{(2\pi)^2} d^2k = \frac{A \cdot 2\pi k dk}{(2\pi)^2} = \frac{A}{(2\pi)} k dk \tag{27}$$

   If we fill up states at $T = 0$ up to the Fermi wavevector k_F, then

$$\begin{aligned} N &= 2\frac{A}{(2\pi)} \int_0^{k_F} k dk \\ &= 2\frac{A}{(2\pi)} \frac{1}{2} k_F^2 \\ &= \frac{A k_F^2}{2\pi} \\ k_F^2 &= \frac{2\pi N}{A} \\ E_F &= \frac{\hbar^2 k_F^2}{2m} \\ &= \frac{2\pi \hbar^2 N}{2mA} \\ &= \frac{\pi \hbar^2 N}{mA} \end{aligned} \tag{28}$$

   where the factor of 2 comes from the 2 spin states of the electron. Now use

$$\begin{aligned} Q &= \Delta N e = CV \\ \Delta N &= \frac{CV}{e} \\ N &= N_o + \Delta N \end{aligned} \tag{29}$$

$$E_F = \frac{\pi \hbar^2}{mA} \left(N_o + \frac{CV}{e} \right) \tag{30}$$

Alternatively, one can go back to Eq. (27) and convert kdk into dE as follows:

$$E = \frac{\hbar^2 k^2}{2m} \qquad dE = \frac{\hbar^2 kdk}{m} \rightarrow kdk = \frac{mdE}{\hbar^2} \tag{31}$$

So the density of states is

$$\rho(E)dE = \frac{A}{2\pi}kdk = \frac{Am}{2\pi\hbar^2}dE \tag{32}$$

Summing over the occupation of all states must yield N.

$$N = 2\sum_r \frac{1}{e^{\beta(E-\mu)}+1} \tag{33}$$

where the factor of 2 is for spin, $\beta = 1/k_BT$, and μ is the chemical potential. Convert the sum into an integral:

$$\begin{aligned} N &= 2\int_0^\infty dE\rho(E)\frac{1}{e^{\beta(E-\mu)}+1} \\ &= \frac{2Am}{2\pi\hbar^2}\int_0^\infty dE\rho(E)\frac{1}{e^{\beta(E-\mu)}+1} \end{aligned} \tag{34}$$

Note that $E_F = \mu(T=0)$. At $T = 0$, $\beta = \infty$ and the Fermi function becomes a step function that is unity for $E < E_F$ and zero otherwise. So we have

$$\begin{aligned} N &= \frac{2Am}{2\pi\hbar^2}\int_0^{E_F} dE \\ &= \frac{Am}{\pi\hbar^2}E_F \\ E_F &= \frac{\pi\hbar^2}{m}\frac{N}{A} \end{aligned} \tag{35}$$

Thus we recover Eq. (28).

$$\boxed{E_F(V) = \frac{\pi\hbar^2}{mA}\left(N_o + \frac{CV}{e}\right)}$$

3. (10 points) Consider an equilibrium system with N identical noninteracting magnetic atoms subjected to an external magnetic field $\vec{H} = H\hat{z}$ at temperature T. Each atom has spin 1/2 and a magnetic moment $\vec{\mu}$. When $\vec{\mu}$ is parallel to $\vec{H}$, the energy is $E_- = -\mu H$. When $\vec{\mu}$ is antiparallel to $\vec{H}$, the energy is $E_+ = +\mu H$. Find the magnetization M/N per spin and the entropy per spin S/N as a function of T and H.

   Solution:

   Start with the partition function Z:

$$\begin{aligned} Z &= \zeta^N \\ \zeta &= \sum_r e^{-\beta E_r} = e^{\beta\mu H} + e^{-\beta\mu H} = 2\cosh(\beta\mu H) = 2\cosh\left(\frac{\mu H}{k_B T}\right) \\ \ln Z &= N \ln\zeta \\ &= N\ln\left[2\cosh\left(\frac{\mu H}{k_B T}\right)\right] \\ F &= -k_B T\ln Z = -Nk_B T\ln\left[2\cosh\left(\frac{\mu H}{k_B T}\right)\right] \qquad (36) \end{aligned}$$

   There are 2 ways to calculate the magnetization per spin M/N:

   (a)

$$\begin{aligned} M &= \frac{1}{\beta}\frac{\partial \ln Z}{\partial H} \\ &= \frac{N}{\beta}\frac{\partial\ln\zeta}{\partial H} \\ \frac{M}{N} &= \mu\ \frac{e^{\beta\mu H} - e^{-\beta\mu H}}{e^{\beta\mu H} + e^{-\beta\mu H}} \\ &= \mu\tanh\left(\frac{\mu H}{k_B T}\right) \qquad (37) \end{aligned}$$

   (b)

$$\begin{aligned} \overline{M} &= N\overline{\mu} \\ &= N\left(P_\uparrow\mu_\uparrow + P_\downarrow\mu_\downarrow\right) \\ &= N\mu\ \frac{e^{\beta\mu H} - e^{-\beta\mu H}}{e^{\beta\mu H} + e^{-\beta\mu H}} \qquad (38) \end{aligned}$$

   where we used $\mu_\uparrow = -\mu_\downarrow = \mu$, $P_\uparrow = \exp(\beta\mu H)/\zeta$, and $P_\downarrow = \exp(-\beta\mu H)/\zeta$. Thus

$$\frac{\overline{M}}{N} = \mu\tanh\left(\frac{\mu H}{k_B T}\right) \qquad (39)$$

There are 2 ways to obtain the entropy S:

(a)

$$\begin{aligned}
S &= k_B\left(\ln Z + \beta\overline{E}\right) \\
\overline{E} &= -\frac{\partial \ln Z}{\partial \beta} = -N\frac{\partial}{\partial\beta}\ln\left(e^{\beta\mu H}+e^{-\beta\mu H}\right) = -N\frac{\left(\mu H e^{\beta\mu H}-\mu H e^{-\beta\mu H}\right)}{e^{\beta\mu H}+e^{-\beta\mu H}} \\
&= -N\mu H\tanh\left(\frac{\mu H}{k_B T}\right) \\
\frac{S}{N} &= k_B\left\{\ln\left[2\cosh\left(\frac{\mu H}{k_B T}\right)\right]-\left(\frac{\mu H}{k_B T}\right)\tanh\left(\frac{\mu H}{k_B T}\right)\right\}
\end{aligned}$$

(b) Use F from Eq. (36) and

$$\begin{aligned}
S &= -\left(\frac{\partial F}{\partial T}\right)_V \\
&= Nk_B\ln\left[2\cosh\left(\frac{\mu H}{k_B T}\right)\right]+Nk_BT\frac{2\sinh\left(\frac{\mu H}{k_BT}\right)\cdot\left(-\frac{\mu H}{k_BT^2}\right)}{2\cosh\left(\frac{\mu H}{k_BT}\right)} \\
\frac{S}{N} &= k_B\left\{\ln\left[2\cosh\left(\frac{\mu H}{k_B T}\right)\right]-\left(\frac{\mu H}{k_B T}\right)\tanh\left(\frac{\mu H}{k_B T}\right)\right\}
\end{aligned}$$

$$\boxed{M/N = \mu\tanh\left(\tfrac{\mu H}{k_BT}\right)}$$

$$\boxed{S/N = k_B\left\{\ln\left[2\cosh\left(\tfrac{\mu H}{k_BT}\right)\right]-\left(\tfrac{\mu H}{k_BT}\right)\tanh\left(\tfrac{\mu H}{k_BT}\right)\right\}}$$

4. (10 points) Blackbody radiation: In studying blackbody radiation, we found that the total energy density $\overline{u}_o(T) = \sigma T^4$ where σ is a constant. What is the entropy S of a gas of photons in equilibrium at temperature T in a box of volume V?

Answer:

Method I:

$$\overline{u}_o(T) = \sigma T^4 \tag{40}$$

So the total energy $\overline{E}$ is given by

$$\begin{aligned} \overline{E} &= \overline{u}_o(T)V \\ &= \sigma T^4 V \end{aligned}$$

The heat capacity is given by

$$C_V = \left.\frac{d\overline{E}}{dT}\right|_V = 4\sigma T^3 V \tag{41}$$

The entropy is given by

$$S = \int_0^T \frac{C_V}{T} dT = 4\sigma V \int_0^T T^2 dT = \frac{4}{3}\sigma T^3 V \tag{42}$$

Method II:

$$\begin{aligned} d\overline{E} &= dQ - dW \\ dW &= 0 \\ d\overline{E} &= dQ = TdS \\ dS &= \frac{d\overline{E}}{T} \end{aligned}$$

Now use

$$\begin{aligned} \overline{E} &= \overline{u}_o(T)V = \sigma T^4 V \\ d\overline{E} &= \frac{d\overline{E}}{dT} dT = 4V\sigma T^3 dT \end{aligned}$$

to obtain

$$\begin{aligned} S &= \int \frac{d\overline{E}}{T} \\ &= \int_0^T \frac{4\sigma V T^3 dT}{T} \\ &= 4\sigma V \int_0^T T^2 dT \\ &= \frac{4}{3}\sigma V T^3 \end{aligned}$$

$$\boxed{S = \frac{4}{3}\sigma T^3 V}$$

5. (10 points) A Metropolis Monte Carlo simulation of a large interacting system is performed at a fixed temperature T and a fixed volume V. The temperature is high in the sense that it is larger than any interaction energies. A histogram is made of the energies that the system has during the course of the simulation. A Gaussian distribution is a good fit to the histogram. Thus, the energy distribution $P(E)$ is given by

$$P(E) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(E-\overline{E})^2/2\sigma^2} \tag{43}$$

where E is the energy of the system, $\overline{E}$ is the mean energy, and σ is the root-mean-square deviation of the energy from the mean energy. Find the heat capacity $C_V(T)$ of this system.

Answer: From lecture 9:

$$C_V(T) = k_B\beta^2\overline{(\Delta E)^2} = \frac{\overline{(\Delta E)^2}}{k_B T^2} = \frac{\sigma^2}{k_B T^2} \tag{44}$$

To see where this comes from, let us calculate the dispersion $\overline{(\Delta E)^2}$ in the energy:

$$\overline{(\Delta E)^2} = \overline{(E-\overline{E})^2} = \overline{E^2 - 2E\overline{E} + \overline{E}^2} = \overline{E^2} - \overline{E}^2 \tag{45}$$

$$\overline{E} = -\frac{\partial \ln Z}{\partial \beta} = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} \tag{46}$$

We need now to compute $\overline{E^2}$:

$$\overline{E^2} = \frac{\sum_r E_r^2 e^{-\beta E_r}}{\sum_r e^{-\beta E_r}} \tag{47}$$

But

$$\sum_r E_r^2 e^{-\beta E_r} = -\frac{\partial}{\partial \beta}\left(\sum_r E_r e^{-\beta E_r}\right) = \left(-\frac{\partial}{\partial \beta}\right)^2 \left(\sum_r e^{-\beta E_r}\right) \tag{48}$$

And from the definition of the partition function Z

$$\overline{E^2} = \frac{1}{Z}\frac{\partial^2 Z}{\partial \beta^2} \tag{49}$$

This can be rewritten as

$$\overline{E^2} = \frac{\partial}{\partial \beta}\left(\frac{1}{Z}\frac{\partial Z}{\partial \beta}\right) + \frac{1}{Z^2}\left(\frac{\partial Z}{\partial \beta}\right)^2 = -\frac{\partial \overline{E}}{\partial \beta} + \overline{E}^2 \tag{50}$$

Finally we obtain

$$\overline{(\Delta E)^2} = \overline{E^2} - \overline{E}^2 = -\frac{\partial \overline{E}}{\partial \beta} \tag{51}$$

or

$$\overline{(\Delta E)^2} = \frac{\partial^2 \ln Z}{\partial \beta^2} \tag{52}$$

Now recall

$$C_V = \left.\frac{\partial \overline{E}}{\partial T}\right|_V = \left(\frac{\partial \overline{E}}{\partial \beta}\frac{\partial \beta}{\partial T}\right)_V = -\left.\frac{1}{k_B T^2}\frac{\partial \overline{E}}{\partial \beta}\right|_V = -\left.k_B \beta^2 \frac{\partial \overline{E}}{\partial \beta}\right|_V \tag{53}$$

But

$$\overline{E} = -\frac{\partial \ln Z}{\partial \beta} \tag{54}$$

Therefore

$$C_V = k_B \beta^2 \frac{\partial^2 \ln Z}{\partial \beta^2} = k_B \beta^2 \overline{(\Delta E)^2} = \frac{\sigma^2}{k_B T^2} \tag{55}$$

$$\boxed{C_V(T) = \tfrac{\sigma^2}{k_B T^2}}$$