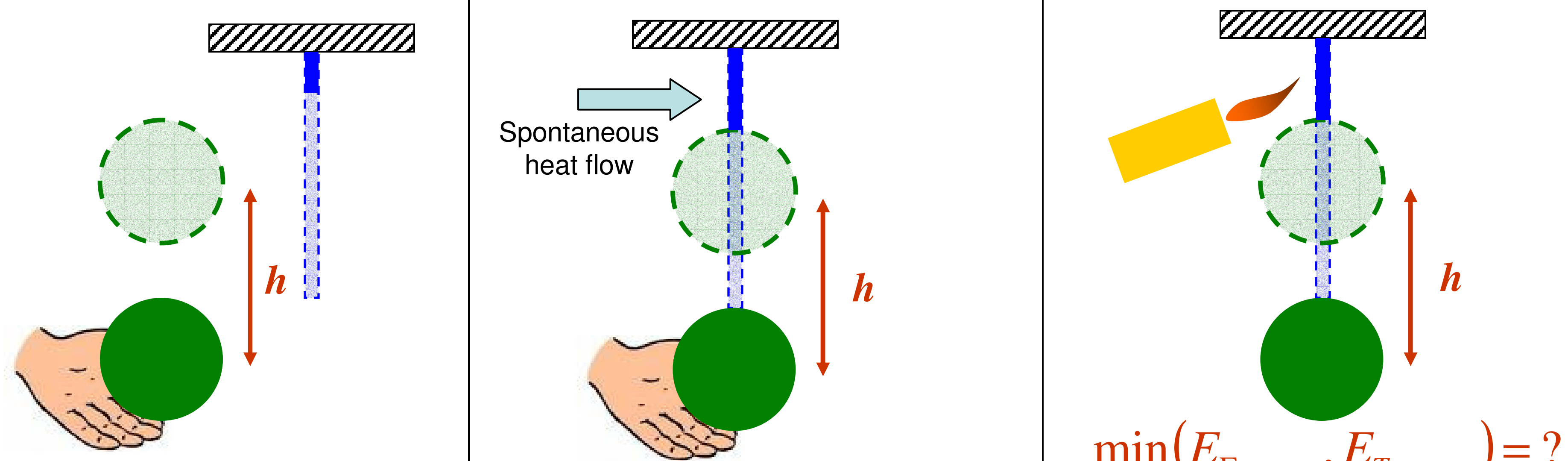
Energy-efficient use of Entropy

Problem: to minimize energy E spent to lift a weight using a stretched rubber band.

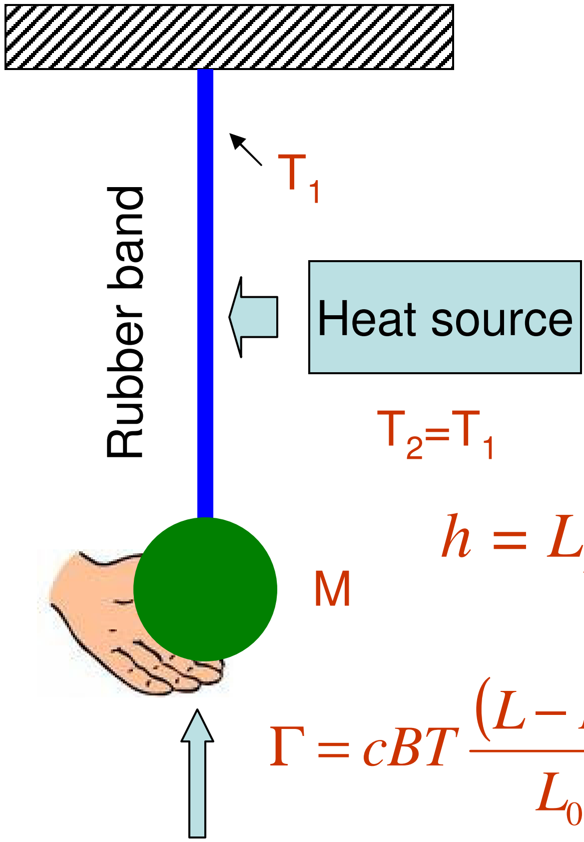
Lift by hand, no help from the rubber band	Lift by hand at constant T, rubber band helps	Lift by warming the rubber band
$E_0 = W = Mgh$	$E_{T=const} = W' = ?$	$E_{\Gamma=const} = Q = ?$

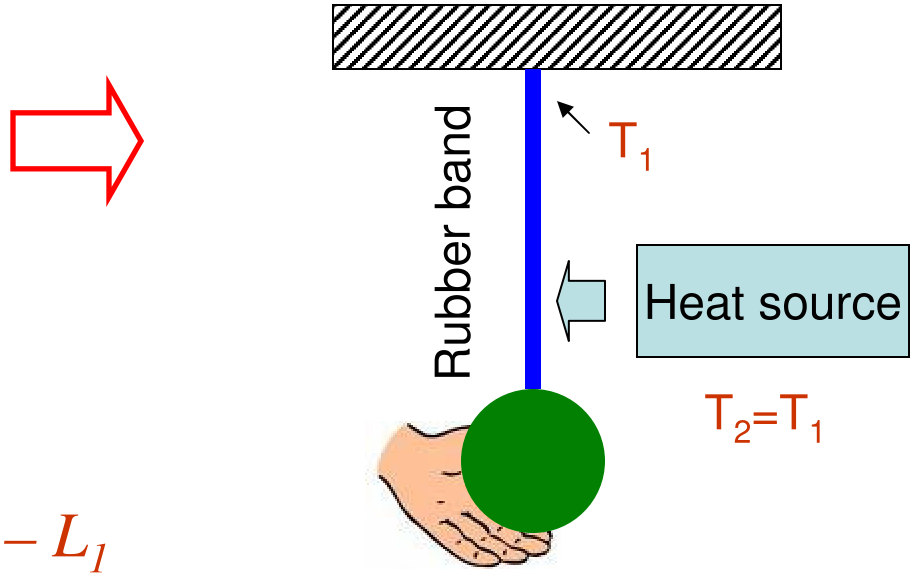


$$\min\left(E_{\Gamma=const}, E_{T=const}\right) = ?$$

Notes

Example of a heat engine: isothermal rubber band





$$h = L_2 - L_1$$

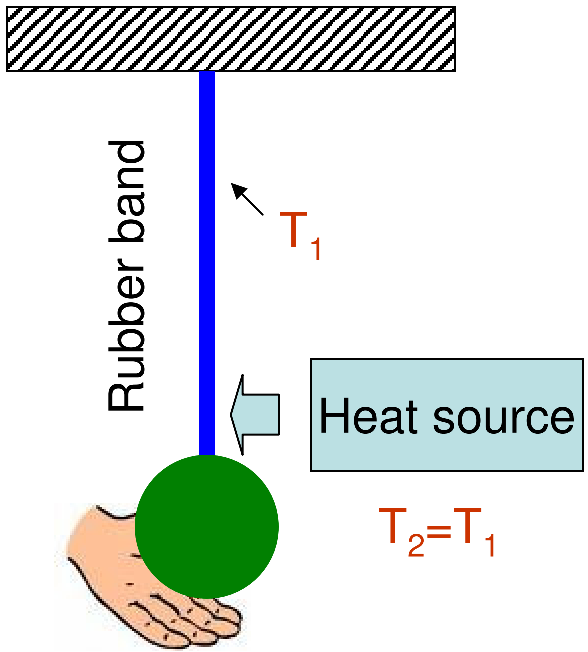
Applying force to the weight so that the rubber band contracts from L_1 to L_2 isothermally

$$\Gamma = cBT\frac{(L-L_0)}{L_0}$$

$$W_{RWS} = -\int_{L_1}^{L_2}\Gamma dL = \frac{cBT}{2L_0}\left((L_1-L_o)^2-(L_2-L_o)^2\right)$$

Notes

Example of a heat engine: isothermal rubber band



Reversible heat engine

$$U = cL_0 T \Longrightarrow \Delta U = 0$$

$$\Delta U = \Delta Q_{RHS} + \Delta W_{RWS} \Longrightarrow \Delta Q_{RHS} = -\Delta W_{RWS}$$

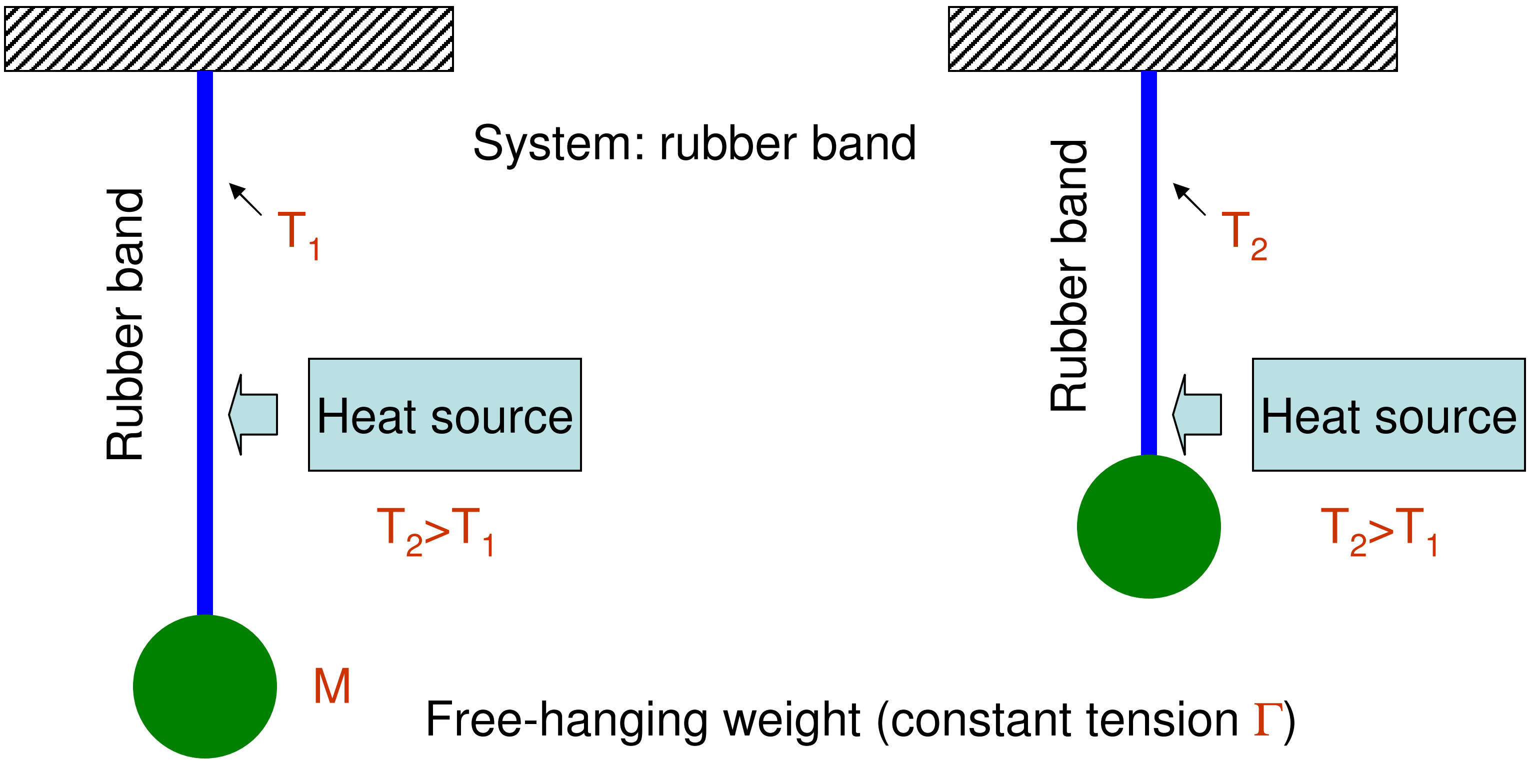
$$S_{band} = cL_0 \ln(U) - \frac{cB}{2}\frac{(L-L_0)^2}{L_0} + const$$

$$\Delta S_{band} = \frac{\Delta W_{RWS}}{T} \qquad \Delta S_{RHS} = \frac{\Delta Q_{RHS}}{T}$$

$$\Delta S_{total} = \Delta S_{band} + \Delta S_{RHS} \Longrightarrow \Delta S_{total} = 0$$

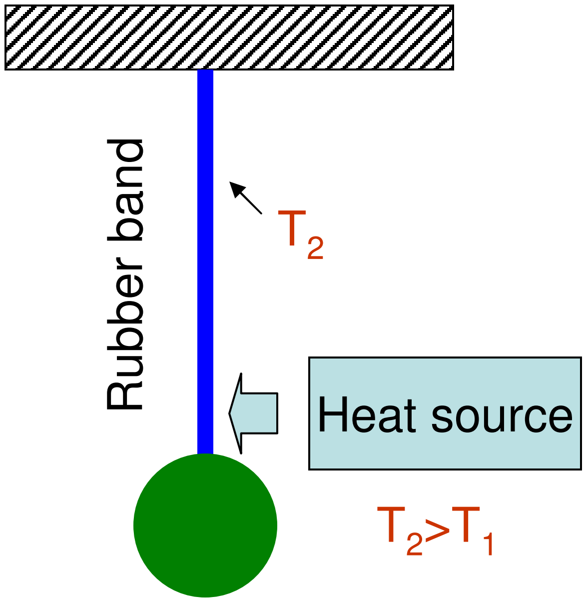
Notes

Example of a heat engine: isobaric rubber band



Notes

Example of a heat engine: isobaric rubber band



Free-hanging weight
(constant tension Γ)

$$U = cL_0T \Rightarrow \Delta U = cL_0\left(T_2 - T_1\right)$$

$$\Gamma = cBT\frac{\left(L - L_0\right)}{L_0} = Mg$$

$$\Downarrow$$

$$h = L_2 - L_1 = \frac{MgL_0}{cB}\left(\frac{1}{T_2} - \frac{1}{T_1}\right)$$

Notes

Example of a heat engine: isobaric rubber band

$$\Delta U = cL_0\left(T_2 - T_1\right)$$

Work done by the rubber band $$\Delta W_{RWS} = Mgh = \frac{M^2 g^2 L_0\left(T_2 - T_1\right)}{cBT_1T_2} > 0$$

Heat transferred from the rubber band

$$\Delta Q_{RHS} = -\Delta U - \Delta W_{RWS} = -cL_0(T_2 - T_1) - Mgh = -cL_0(T_2 - T_1)\left(1 + \frac{M^2 g^2}{c^2 BT_1T_2}\right) < 0$$

Heat transferred to the rubber band

$$E_{\Gamma=const} = Q = \left|\Delta Q_{RHS}\right| = cL_0(T_2 - T_1) + Mgh > Mgh$$

Notes

Example of a heat engine: isobaric rubber band

$$S = cL_0 \ln(U) - \frac{cB}{2}\frac{(L-L_0)^2}{L_0} + const$$

Change of entropy of the band:

$$\Delta S = cL_0 \ln\left(\frac{T_2}{T_1}\right) - \frac{cB}{2L_0}\left((L_2-L_0)^2 - (L_1-L_0)^2\right) =$$

$$= cL_0 \ln\left(\frac{T_2}{T_1}\right) + \frac{cB}{2L_0}\left(\frac{MgL_0}{cB}\right)^2 \frac{(T_2-T_1)(T_2+T_1)}{T_1^2 T_2^2}$$

$$\boxed{\Delta S = cL_0 \ln\left(\frac{T_2}{T_1}\right) + \frac{cB}{2L_0T_2^2}\left(\frac{MgL_0}{cB}\right)^2\left(\frac{T_2^2}{T_1^2} - 1\right)}$$

Notes

Example of a heat engine: isobaric rubber band

Change of entropy of the heat source:

$$\Delta S_{RHS} = \frac{\Delta Q_{RHS}}{T_2} = -cL_0\left(\frac{T_2 - T_1}{T_2}\right)\left(1 + \frac{M^2 g^2}{c^2 B T_1 T_2}\right)$$

Total entropy change of the system: $\Delta S_{total} = \Delta S_{RHS} + \Delta S$

$$\Delta S_{total} = -cL_0 \frac{(T_2 - T_1)}{T_2}\left(1 + \frac{M^2 g^2}{c^2 B T_1 T_2}\right) + cL_0 \ln\left(\frac{T_2}{T_1}\right) + \frac{cB}{2L_0 T_2^2}\left(\frac{MgL_0}{cB}\right)^2\left(\frac{T_2^2}{T_1^2} - 1\right)$$

$$\Delta S_{total} = cL_0 \ln\left(\frac{T_2}{T_1}\right) - cL_0\left(1 - \frac{T_1}{T_2}\right) + \frac{M^2 g^2 L_0}{2cBT_2^2}\left(\frac{T_2}{T_1} - 1\right)^2$$

Notes

Example of a heat engine: isobaric rubber band

$$\Delta S_{total} = cL_0 \ln\left(\frac{T_2}{T_1}\right) - cL_0\left(1 - \frac{T_1}{T_2}\right) + \frac{M^2 g^2 L_0}{2cBT_2^2}\left(\frac{T_2}{T_1} - 1\right)^2$$

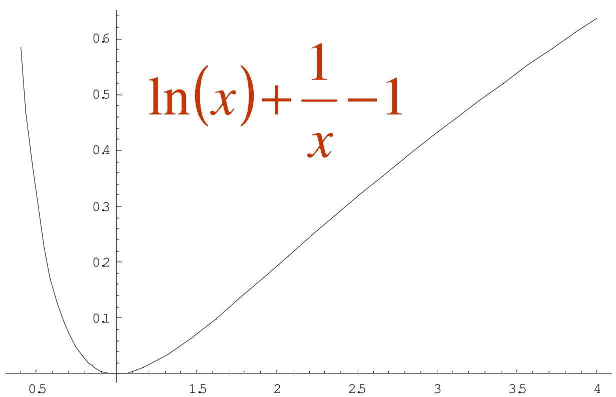
$\Delta S_{total} \geq 0$ because $\frac{M^2 g^2 L_0}{2cBT_2^2}\left(\frac{T_2}{T_1} - 1\right)^2 \geq 0$ and $cL_0 \ln\left(\frac{T_2}{T_1}\right) - cL_0\left(1 - \frac{T_1}{T_2}\right) \geq 0$

The latter is equivalent to:

$$\ln(x) + \frac{1}{x} \geq 1$$

Which is a true statement

Plot[Log[x]+1/x−1,{x,0.4,4},PlotRange->All]



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$$\Delta S_{total} > 0$$

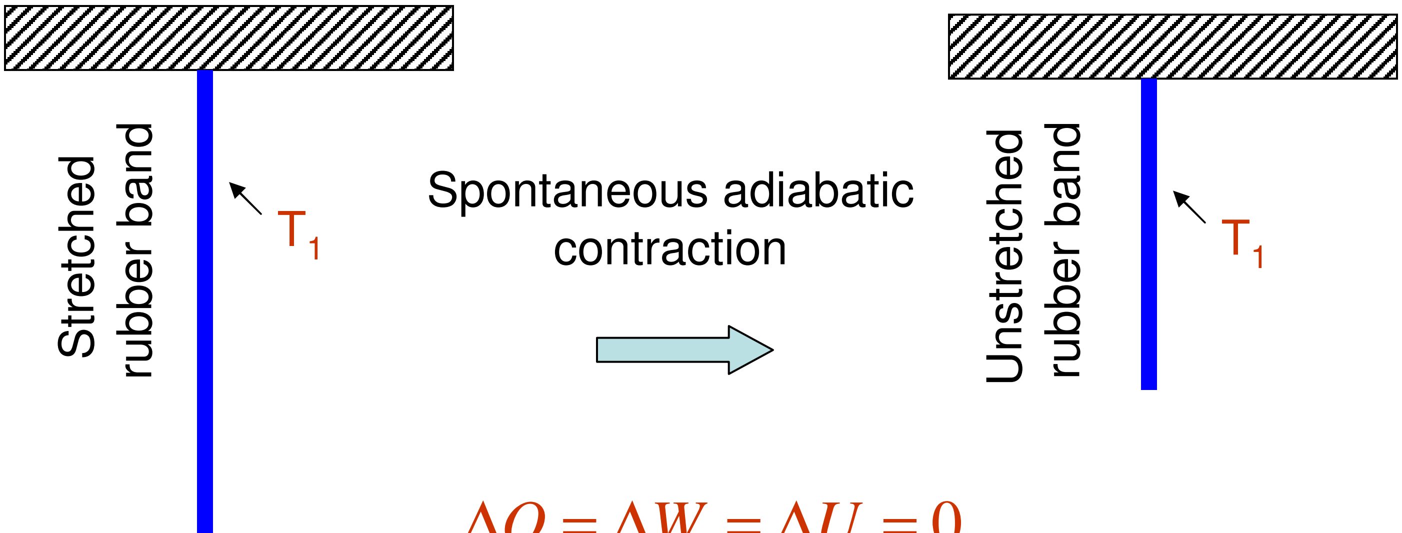
If $T_2 \neq T_1$

Isobaric rubber band engine is irreversible!

Notes

Harvesting Low Entropy to do work: rubber band

Case 1 – low entropy wasted



$$\Delta Q = \Delta W = \Delta U = 0$$

Notes

Harvesting Low Entropy to do work: rubber band

$$\Delta Q = \Delta W = \Delta U = 0$$

Initial entropy

$$S_i = cL_0 \ln(U) - \frac{cB}{2} \frac{(L - L_0)^2}{L_0} + const$$

final entropy

$$S_f = cL_0 \ln(U) + const$$

$$\boxed{\Delta S_{total} = \Delta S = S_f - S_i = \frac{cB}{2} \frac{(L - L_0)^2}{L_0} > 0}$$

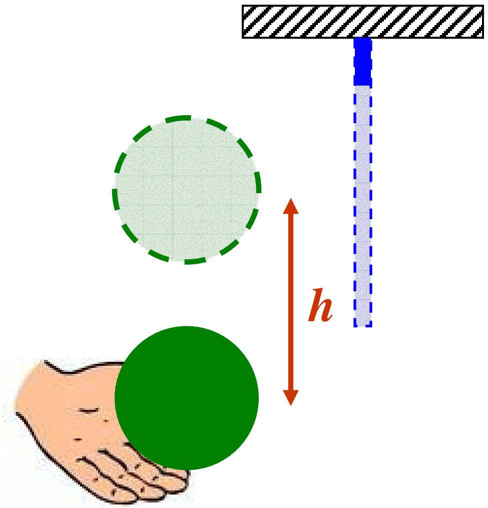
No work done, total entropy increased

Notes

Summary

Lift by hand, no help from the rubber band

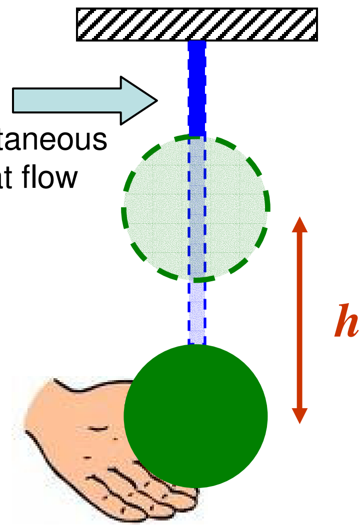
$$E_0 = W = Mgh$$



$$\Delta S_{total} > 0$$

Lift by hand at constant T, rubber band helps

$$E_{T=const} = W < Mgh$$



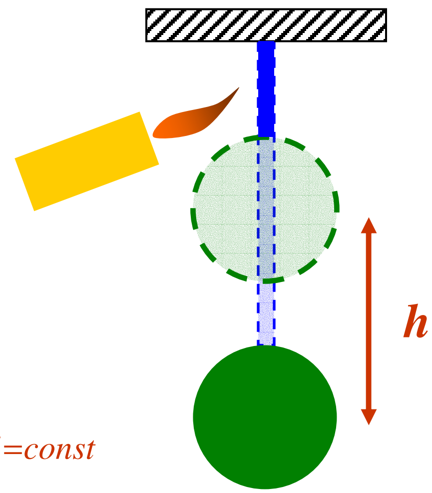
$$\Delta S_{total} = 0$$

$$E_{\min} = E_{T=const}$$

Least energy is spent in the reversible process to lift the weight

Lift by warming the rubber band

$$E_{\Gamma=const} = Q = Mgh + \Delta U$$



$$\Delta S_{total} > 0$$

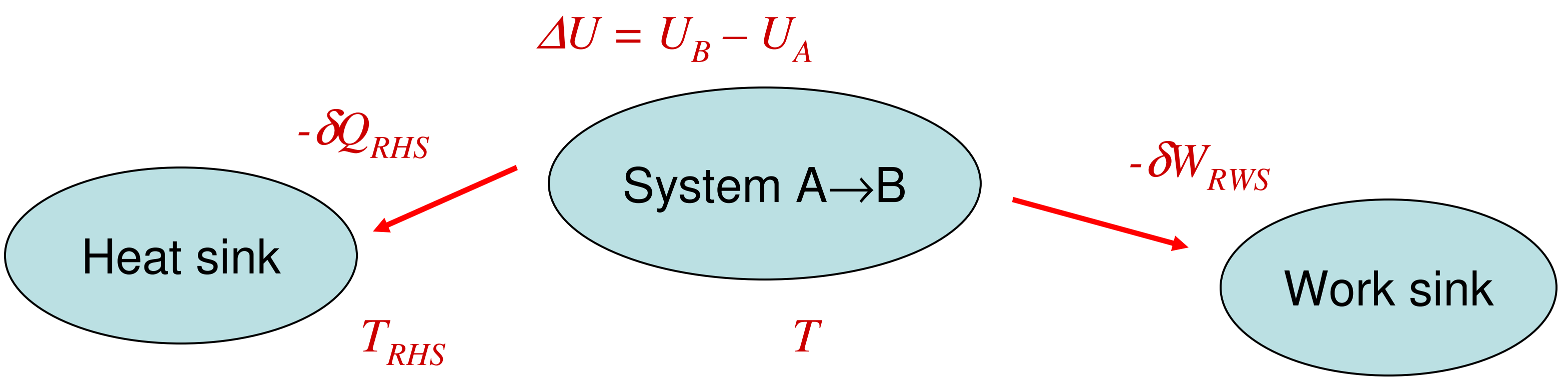
Notes

The Maximum Work Theorem

We model the environment of the system as two objects:

- reversible work source (sink) RWS
- reversible heat source (sink) RHS

Heat engine model



Rigid walls, so no work can be done on the heat sink !

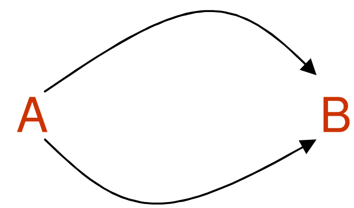
Adiabatic walls, so no heat can be transferred to the work sink! ($dS_{RWS}=0$)

Notes

The Maximum Work Theorem

<u>Maximum work theorem:</u> for a given ΔU, $-\delta W$ is maximum for a reversible process

System goes from state A to state B.
Which process delivers maximum work
to the reversible work source?



The change of internal energy ΔU is the same for all $A \rightarrow B$ processes

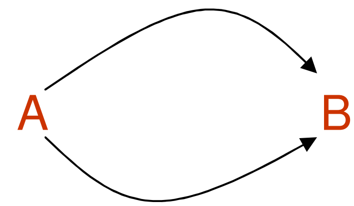
ΔU is distributed between work done on RWS, ΔW, and heat flow to the RHS, ΔQ

Therefore, maximum work delivered to RWS implies minimum heat flow to RHS.

Notes

The Maximum Work Theorem

<u>Maximum work theorem:</u> for a given ΔU, $-\delta W$ is maximum for a reversible process



The change of entropy of the working body ΔS is the same for all $A \rightarrow B$ processes

The change of entropy of the RHS is proportional to ΔQ, therefore maximum work corresponds to the minimum entropy change of the RHS and (since ΔS is independent on the process $A \rightarrow B$) of the entire composite system (engine).

The absolute minimum entropy increase is zero and achieved in a reversible process.

Therefore, maximum work done by a heat engine is achieved in a reversible process.

Notes

The Maximum Work Theorem

<u>Maximum work theorem:</u> for a given ΔU, $-\delta W$ is maximum for a reversible process

Proof:

$$dS_{total} = dS + dS_{RHS} = dS + \frac{\delta Q_{RHS}}{T_{RHS}} \geq 0 \Longrightarrow \delta Q_{RHS} \geq -T_{RHS}dS$$

$$dU = -\delta W_{RWS} - \delta Q_{RHS}$$

$$\Downarrow$$

$$dU \leq -\delta W_{RWS} + T_{RHS}dS$$

$$\Downarrow$$

$$\boxed{\delta W_{RWS} \leq T_{RHS}dS - dU}$$

Functions of states A and B only, not of the process A→B

Notes

The Maximum Work Theorem

$$\boxed{\delta W_{RWS} \le T_{RHS} dS - dU}$$

The work is maximum when $\le \to =$

$$\delta W_{RWS}^{\max} = T_{RHS} dS - dU$$

The origin of the $\le$ sign is traced back to $dS_{total} \geq 0$.
Therefore, for maximum work we have $dS_{total} = 0$,
hence reversible process maximizes work.

Notes

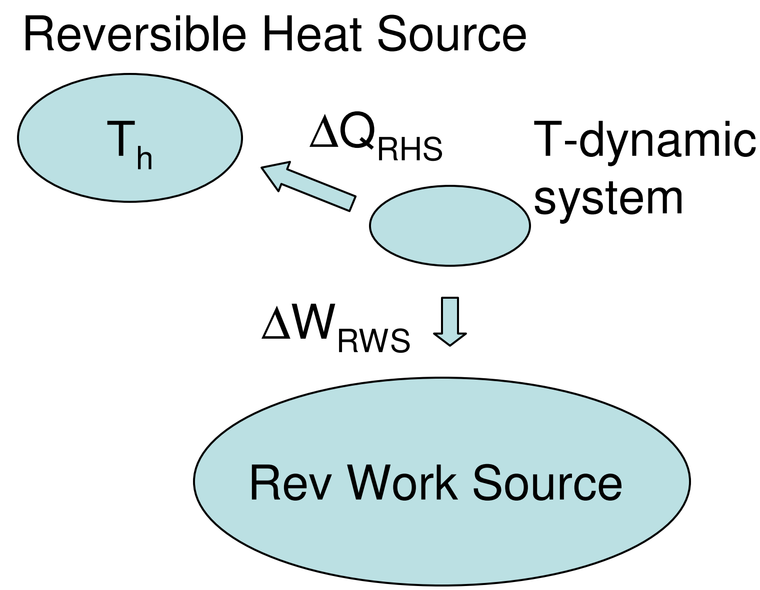
Summary of heat engines so far

- A thermodynamic system in a process connecting state A to state B

- In this process, the system can do work and emit/absorb heat

- What processes maximize work done by the system?

- We have proven that reversible processes (processes that do not change the total entropy of the system and its environment) maximize work done by the system in the process $A \rightarrow B$



Process $A \rightarrow B$ of the system

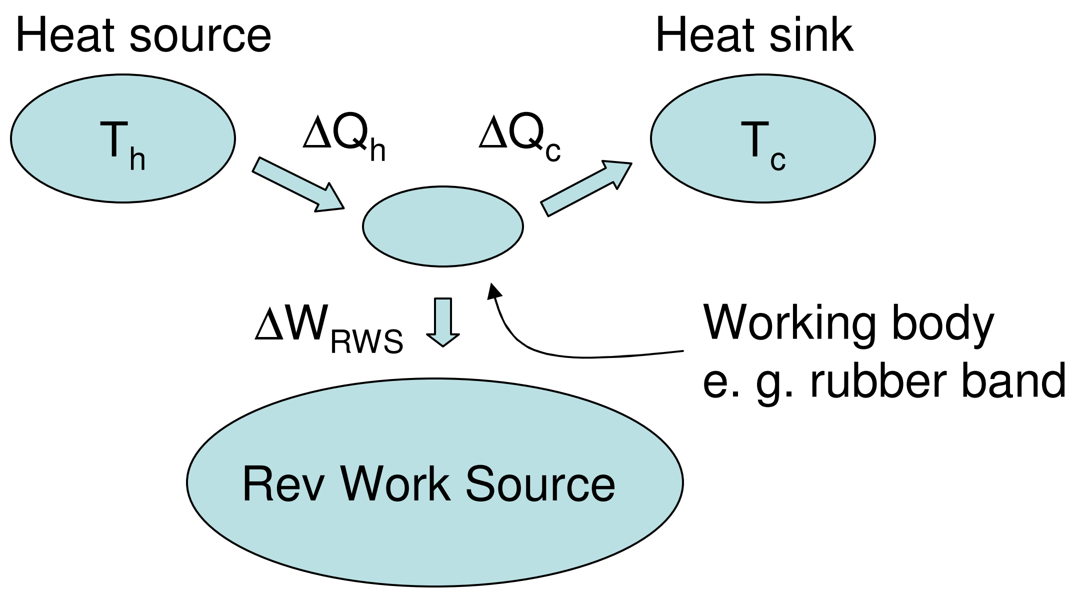
Notes

Cyclic Engine

-To continuously generate work, a heat engine should go through a cyclic process.

- Such an engine needs to have a heat source and a heat sink at different temperatures.

- The net result of each cycle is to extract heat from the heat source and distribute it between the heat transferred to the sink and useful work.



Notes